Efficient Collusion in Sponsored Search Auctions

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by

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The challenge of innovation is that we are all boxed in by what we know, by our assumptions about how things work. Innovation is right in front of us—we just need to see past our own assumptions. Forget what you know.

—Scott Karp on Google's *Adwords Select*—

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1 Introduction

The online advertisement market is one of the most blossoming and rapidly growing markets today and billions of dollars worth of keywords are sold every year. The revenue of the major player in the market, Google, added up to about $ 4.226 billion in 2008. On the Forbes Global 2000[1], a list of the 2000 largest companies in the world, Google was ranked in position 155. In order to demonstrate the immense dimensions of the Internet advertisement market, it is found in an earlier version of (Edelman, Ostrovsky, Schwarz [2007]) from October 2005:

> "*The combined market capitalization of Google and Yahoo! is over $ 125 billion. In comparison, the combined market capitalization of all US airlines is about $ 20 billion.*"

Today, the market capitalization of Google alone is about $ 125.3 billion. The dominant part of sales in the Internet advertising market is organized by an auction mechanism, the so-called *generalized second-price auction (GSP)*, that was introduced in its current version not earlier than February 2002 by Google. This mechanism is subject of our study. We will investigate the strategic implications of this mechanism, i.e. we will investigate the set of Nash equilibria; introduce the notion of mediators and question whether it is vulnerable to collusion by bidders who use a mediation device in order to do so.

The mechanism works as follows: Whenever an Internet user enters a query, she will be redirected to a page of results, on which she finds a list of so-called *sponsored links* next to the list of results, which are clearly distinguishable from the ordinary search

[1] `http://www.forbes.com/lists/2009/18/global-09_The-Global-2000_Rank_print.html`

results. These sponsored links are paid advertisements, targeted to the specific keywords that users enter. When a user clicks on a link, she will be redirected to the respective advertiser's page. The search engine charges a certain amount of money from each advertiser for every click on his advertisement depending on the keyword's popularity, as well as for the position in which he is shown. In particular, there is a limited number of ads that can be shown on a page, and links that are shown on a higher position on the screen are likely to be clicked more often. Advertisers submit one-dimensional bids for each keyword they are interested in and, finally, they are shown in decreasing order of their bids whenever the respective keyword is entered by a user. The auction is highly dynamic, since bidders can adjust their bids at any point in time, and different links are shown for different keywords. Each advertiser has to pay the amount of the bid that the advertiser has submitted that is allocated to the position directly below him plus a minimal increment. The auction is called a *generalized second-price auction* since it is an intuitive extension of the principle of the well known single-item second-price auction. If there were only one position, the mechanism would be strategically equivalent to the famous Vickrey-Clarke-Groves (VCG) auction. However, since there is, in general, more than one position, GSP is no longer equivalent to the VCG auction. It lacks some of the desirable properties such as incentive-compatibility, i.e. truth-telling is a weakly dominant strategy for every bidder. In addition, it does not possess an equilibrium in dominant strategies neither.

In particular, we will analyze the set of Nash equilibria of the game induced by the generalized second-price auction. It can be shown that multiple equilibria exist, and among the set of symmetric Nash equilibria, an equilibrium exists with the same outcome as if the auction were designed according to the rules of VCG. Players are ranked according to their true valuations and they are charged VCG payments. This is the socially optimal outcome for the bidders and the worst outcome for the search engine, which is somewhat surprising, since, the opposite is true of general multiple object auctions, see for example (Krishna [2002]):

"*Among all mechanisms for allocating multiple objects that are efficient, in-*

centive compatible and individually rational, the VCG mechanism maximizes the expected payment of each agent."

We will derive our results by following two different approaches—those of (Varian [2007]), who analyzes bounds of equilibrium bids in a symmetric Nash equilibrium, and of (Edelman, Ostrovsky, Schwarz [2007]), who develop the concept of locally envy-free equilibria. In a locally envy-free equilibrium, every bidder prefers his current position to the position of the player directly above him. In fact, we will show that both equilibrium concepts deliver the same results since they are strategically equivalent.

However, as there are multiple equilibria, and from the perspective of the bidders, the VCG equivalent outcome is socially optimal, bidders can improve their utility by coordinating to the VCG equilibrium over any other equilibrium of the game. (Edelman, Ostrovsky, Schwarz [2007]) show by means of a *generalized English auction* (that corresponds strategically to the generalized second-price auction) that bidders may emerge to coordinate with the VCG outcome in a dynamic framework through simple strategies, i.e. by raising their bids incrementally until it is no longer profitable. Another way to attain the VCG equivalent equilibrium is by the use of a mediator. In the thesis at hand, we will focus on mediators in terms of (Ashlagi et al. [2008]):

> "*A mediator for a given game is a reliable entity that can interact with the players and perform actions on their behalf. However, a mediator cannot enforce behavior.*"

Every player is free to use the service of the mediator or participate in the game directly. (Ashlagi et al. [2008]) design a mediator that implements the VCG equivalent outcome such that it is an ex post equilibrium for every bidder to use the service of the mediator and report his valuation truthfully. We will present their results in some detail in Chapter 4. While a mediator that implements the VCG equivalent outcome may already be regarded as a form of collusive behavior on the part of the bidders, since it coordinates on the best social equilibrium for the bidders and the worst for the search engine, we examine if a mediator exists that is able to coordinate the bidders to achieve even more profitable, collusive outcomes. Our focus is on efficient collusion, i.e. collusive agreements that implement an allocation of players and positions that is consistent with the order of players' true valuations. We find that no such mediator in the fashion of (Ashlagi et al. [2008]) is able to do so. Furthermore, we question if collusion becomes feasible by equipping the mediator with the additional ability to facilitate transfer payments among bidders. In our analysis, we allow for static as well as repeated play.

In Chapter 2 we give a short review on the evolution of sponsored search auctions and introduce a formal model of position auctions, as we will say in place of sponsored search auctions in the remainder of this thesis, since it more accurately depicts the nature of the auctions, i.e. assigning players to positions on a screen. Chapter 3 analyzes equilibrium behavior and outcomes in such a framework. Chapter 4 introduces the concept of mediators and applies it to the context of position auctions. A mediator is presented that implements VCG in a generalized second-price auction. Chapter 5 investigates GSP's vulnerability to collusion. In Chapter 6, we present some variations of our model. Chapter 7 concludes.

2 Position Auctions

2.1 Evolution

The story of Internet advertising began around the year 1994. In the early days of Internet advertising, the predominant format of advertisements was the so-called *banner ads*. These were sold on a CPM (*cost per thousand impressions*) basis, i.e. advertisers paid a fixed amount of money for a fixed number of times their ad was shown, typically one thousand impressions. Single contracts were negotiated on a case-by-case basis, market dynamics were at most moderate and revenues were insignificant compared to what would follow some years later. But in 1997, the Internet advertisement market was revolutionized by GoTo, which was renamed to Overture some time later and was finally acquired by Yahoo!. A completely new way of selling advertisements on the web was created. Not only did advertisers no longer have to pay for a fixed number of impressions anymore, but they were also charged a price per click, i.e. for every time a user was redirected from a search engine to their respective web site. Furthermore, advertisements became targeted. Each visitor to a web site was no longer shown the same banner, advertisements were now shown to a user whenever she entered an appropriate keyword into a search query. Despite these remarkable features, the new mechanism brought considerable innovation in many more ways. First, it served as a compromise in terms of what was essentially sold. Advertisers are in fact interested in attracting customers who finally purchase a product, while the search engine's goal is to maximize the profit for every query that is performed by a user. While the former suggests a pricing model in which a payment has to be made only if a customer actually purchases a product, the latter suggest cost per impression pricing. Furthermore, the mechanism overcame manual, cost-intensive negotiations for each advertisement by introducing a kind of automated self-serve interface on an auction basis. Until the first years of the new millennium, Overture, which implemented pay-per-click advertisement for other web sites, significantly outperformed Google in terms of revenue by an amount of $ 288 million in 2001, while Google earned about $ 86.4 million in the same year. The initial auction format by Overture was a first-price format, i.e. bidders were charged to pay the bid they submitted on a per-click basis. But this pricing rule turned out to be inefficient. Due to the dynamic nature of ad auctions, bidder's are free to change their bid whenever they want to, it occurred that bidders adjusted their bids frequently in response to other bidders' behavior. In particular, as shown e.g. in (Edelman, Ostrovsky [2005]), the first-price format used by Overture did not possess a stable equilibrium, and "*cycling*" bids were observed in practice. Bidders subsequently raised their bids by an increment above competitor's bids until it was no longer profitable and then dropped back to the minimum bid, which yielded inefficiency for both the search engine and the bidders. Inefficiency from the perspective of the bidders, since the bidder with the highest valuation did not obtain the highest position over the whole period of time, and inefficiency for the search engine, since bids were below equilibrium bids of alternative pricing mechanisms over considerable periods of time. Hence the mechanism caused inefficient allocations, created volatile prices and offered incentives to invest in automated robots and other software that would offer bidders advantages over other bidders who could not react as quickly on changing bids. Clearly, such investments are socially inefficient. Finally, in February 2002, Google introduced a new version of its ad program *AdWords*, that initially sold ads on CPM basis. *AdWords Select* came along with a considerable number of innovations that were destined to once more change the market fundamentally. In order to overcome the inefficiencies due to cycling bids, Google introduced a second-price format which was less susceptible to gaming. Another seminal innovation was the introduction of *relevance*. Whenever bidders are ranked according to their bid only, a bidder can obtain the top position on the screen simply by submitting the highest bid. However, if some bidders attract more users than others to click on a link and thus generate higher profits for the search engine, but are assigned to lower positions or not shown at all, this is clearly inefficient. Hence, Google included these bidder dependent probabilities of being clicked by multiplying each bidders bid with its estimated *click-through rate* and ranked the bidders accordingly. While Google remains true to this model to this day, Yahoo! switched to the second-price format shortly after it was introduced by Google, but still ranks bidders purely by their bids. Today, Google earns a revenue of approximately \$ 22 billion, mostly from selling ads. Revenue of Yahoo! aggregates to about \$ 7.2 billion annually.[1]

We will introduce a formal model of position auctions in the following chapter. We will assume the estimated click-through rates to be identical for each bidder, in which case both models are equivalent. In Chapter 6 we will relax this assumption and allow for varying click-through rates and show that most of our forthcoming findings will emerge to hold in adjusted versions.

2.2 Model Environment

In the existing literature, position auctions were modeled as games of complete information by (Edelman, Ostrovsky, Schwarz [2007]) and (Varian [2007]) as well as games of incomplete information, see for example (Ashlagi et al. [2008]) or (Lahaie [2006]). Independent of the information setting that will be applied, position auctions share an identical basic environment $\mathbb{E}_P$.

For a specific keyword, there is a finite number of positions $j \in K = \{1, \ldots, m\}$, i.e. positions on the screen where ads related to the keyword can be displayed. There is a finite number of bidders $i \in N = \{1, \ldots, n\}$, and without loss of generality, there are more bidders than slots, $n > k$. For each position $j \in K$, there exists a positive number $\alpha_j > 0$, the *click-through rate* of position j. In the following, it is assumed

[1]Revenue numbers are taken from www.wolframalpha.com, which emerges to be an exciting new competitor of established search engines.

that the number of clicks received on position j is independent of the identity of the bidder, and that it decreases the lower an ad is placed on the page, i.e. $\alpha_1 > \alpha_2 > \ldots > \alpha_k > 0$. In Section 6.3, we will relax this assumption and show that our analysis remains qualitatively unaffected. In particular, we will assume that the click-through rate of a given position can be factored into a position-specific click-through rate α_j and a bidder-specific factor β_i which we refer to as a bidders' *relevance*. The click-through rate of bidder i assigned to position j is then determined by $\beta_i \alpha_j$. For now, we assume $\beta_i = 1$ for all $i \in N$. The vector $\alpha = (\alpha_1, \ldots, \alpha_k)$ is called the position vector. Every click on an ad of advertiser i shown on position j gives i a revenue of $v_i > 0$, the *valuation* of i. v_i is independent of the position j. The payoff of player i with valuation v_i who is assigned to position j and pays p_j per click then becomes $\omega_i = \alpha_j(v_i - p_j)$. Let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ be the vector of individual payoffs. Each bidder has to submit a bid $b_i \in B_i = [0, \infty)$. Let $\mathbf{B} = B_1 \times B_2 \times \ldots B_n$ be the set of bid profiles. We say that two bids b_i, b_j are distinct if $b_i \neq b_j$. In the position auctions considered here, the bidder with the jth highest bid is assigned to position $j \in K$. If a bidder submits a bid $b_i > 0$ that is not among the m highest bids, he is assigned to a dummy position $k > m$ respectively. We will denote by $b_{(j)}$ the bid of the player who is assigned to position j. We further assume that by bidding $b_i = 0$, a bidder signals that he will not participate in the auction. He is not assigned to any real position and pays 0. For matter of notational convenience, we say he is assigned to the dummy position -1. We define $\alpha_k = \alpha_{-1} = 0$ for all $k > m$. Let us define another dummy position m^+, and let us denote by m^+ the set players that are not assigned to a real position anymore, i.e. to a position $k > m$. Only one player can be assigned to each position $j \in K$, while more than one player can be assigned to the dummy positions -1 and m^+ respectively. Formally, an *allocation* is an assignment of players to positions, $s = (s_1, s_2, \ldots, s_n)$ such that $s_i \neq s_l$ for every $i, l \in K$. $s_i = s(i, \mathbf{b}) = j$ denotes the position of player i given the vector of bids submitted is $\mathbf{b}$. Let $\mathbf{A}$ be the set of all allocations. Given the general environment $\mathbb{E}_P = \{N, K, \alpha, \omega\}$, a position auction is defined by its payment scheme and its tie-breaking rules for the case that there are non-distinct bids $b_i = b_j$.

2.3 Payment Schemes

Given the set of bid profiles $\mathbf{B} = B_1 \times B_2 \times \ldots B_n$, there exists a payment function $p_j : \mathbf{B} \rightarrow \mathbb{R}_+$ for each position $j \in K \cup \{-1, m^+\}$. $p_j(\mathbf{b})$ denotes the payment per click for position j given the bid profile $\mathbf{b}$. Since a bidder who is not assigned to a real position, as well as bidders who do not participate in the auction, pay zero, $p_{m^+} = p_{-1} = 0$. This holds for all position auctions considered in this thesis. Instead, the payment functions $p_j(\mathbf{b})$ differ for all $j \in K$ dependent on the auction format. In a *generalized second-price auction*, let $\forall\, j \in K$ and $\forall\, \mathbf{b} \in \mathbf{B}$

$$p_j^{GSP}(\mathbf{b}) = b_{(j+1)}. \tag{2.1}$$

Every player assigned to a position in K pays the bid of the player assigned to the position directly below him.

LEMMA 1 *Truth telling is not a dominant strategy in the generalized second-price auction.*

Proof. Suppose there are three players and two positions. Let the vector of valuations be $\mathbf{v} = (100, 90, 10)$ and the position vector be $\alpha = (100, 50)$. Given players two and three bid truthfully, the profit of player one given he also submits his bid truthfully would be $\alpha_1 \cdot (v_1 - b_2) = 100 \cdot (100 - 90) = 1000$. By bidding slightly below the bid of player two, say $b_1 = 89$, the payoff of player one would be much higher, i.e. $50 \cdot (100 - 10) = 4500$. Therefore, bidding truthfully is not an equilibrium strategy. $\square$

In contrast, in a *Vickrey-Clarke-Groves (VCG) position auction*, the payment per click of player i assigned to position j depends not only on the bid of the player directly below him, but on the bids of all players l assigned to a position $s(l, \mathbf{b}) > s(i, \mathbf{b})$ as follows:

DEFINITION 1 *Given the environment $\mathbb{E}_P$, then $\forall\, j \in K$ and $\forall\, \mathbf{b} \in \mathbf{B}$, the payment per click of a player assigned to position j induced by the standard Vickrey-Clarke-Groves mechanism is given by*

$$p_j^{VCG}(\mathbf{b}) = \frac{\sum_{k=j+1}^{m+1} b_{(k)}(\alpha_{k-1} - \alpha_k)}{\alpha_j}. \tag{2.2}$$

Proposition 1 *Truth-telling is a weakly dominant strategy under VCG.*

Proof. In order to show that truth-telling is a dominant strategy, suppose there is a mediator that announces he will choose an allocation s^* such that the sum of the reported utilities of the players $i = 1, \ldots, n$ will be maximized. Let the true valuation of player i be v_i and its reported value be b_i. The mediator then announces to pay each agent the sum of the utilities reported by the other players at the utility-maximizing outcome, i.e.

$$b_i \cdot \alpha_{s^*(i,\mathbf{b})} + \sum_{l \neq i} b_l \cdot \alpha_{s^*(l,\mathbf{b})}.$$

Since every player i cares about

$$v_i \cdot \alpha_{s^*(i,\mathbf{b})} + \sum_{l \neq i} b_l \cdot \alpha_{s^*(l,\mathbf{b})}$$

it is obviously a dominant strategy for every player to report $b_i = v_i$. In order to reduce side payments, the mediator subtracts an amount that does not depend on the report of player i and therefore does not bias incentives. He chooses an allocation s^*_{-i} omitting i's report such that

$$s^*_{-i} = \arg\max_{s_{-i}} \sum_{l \neq i} b_l \cdot \alpha_{s_{-i}(l,\mathbf{b})}.$$

The final payoff of player i then becomes

$$v_i \cdot \alpha_{s^*(i,\mathbf{b})} + \sum_{l \neq i} b_l \cdot \alpha_{s^*(l,\mathbf{b})} - \sum_{l \neq i} b_l \cdot \alpha_{s^*_{-i}(l,\mathbf{b})}. \tag{2.3}$$

Rearranging the transfer payments in (2.3) yields (2.2). The net transfer every player i assigned to position $j \in K$ faces can be interpreted as the negative externality he imposes on others. Every player $l \neq i$ assigned to a position $s(l, \mathbf{b}) < s(i, \mathbf{b})$ is not affected by the presence of player i, while every player l assigned to a position $s(l, \mathbf{b}) > s(i, \mathbf{b})$ would arise by one position if player i was absent. □

Moreover, let us state some additional properties of the VCG payment scheme at this point, since it is thematically consistent. We will refer to those in subsequent chapters. The following Lemma and proof is due to (Ashlagi et al. [2008]):

Lemma 2 *Let* $\mathbf{p}^{VCG}$ *be the VCG payment scheme. It holds that*

1. $p_j^{VCG}(\mathbf{b}) \leq b_{(j+1)}$ *for every* $j \in K$.

2. $p_j^{VCG}(\mathbf{b}) \geq p_{j+1}^{VCG}(\mathbf{b})$ *for every* $j = 1, \ldots, m-1$ *and for every* $\mathbf{b} \in \mathbf{B}$*, where for every* j*, equality holds if and only if* $b_{(j+1)} = b_{(j+2)} = \ldots = b_{(m+1)}$.

Proof. Since the VCG payment $p_j^{VCG}(\mathbf{b})$ as defined in (2.2) is a convex combination of the bids of players assigned to positions $j+1, j+2\ldots, m+1$, it never exceeds the maximal element in the sequence, i.e. $b_{(j+1)}$. This proves the first part. In order to prove the second part, note that if $j = m$, then for every $\mathbf{b} \in \mathbf{B}$ it holds that

$$p_j^{VCG}(\mathbf{b}) = b_{(j+1)} \geq 0 = p_{j+1}^{VCG}(\mathbf{b}).$$

For $j < m$, which implies that $j+1 \in K$, and since $b_{(j+1)} \geq b_{(j+2)}$ it holds that

$$\begin{aligned} p_j^{VCG}(\mathbf{b}) &= \frac{b_{(j+1)}(\alpha_j - \alpha_{j+1})}{\alpha_j} + \frac{\sum_{k=j+2}^{m+1} b_{(k)}(\alpha_{k-1} - \alpha_k)}{\alpha_j} \\ &\geq \frac{b_{(j+2)}(\alpha_j - \alpha_{(j+1)})}{\alpha_j} + \frac{\sum_{k=j+2}^{m+1} b_{(k)}(\alpha_{k-1} - \alpha_k)}{\alpha_j}. \end{aligned} \tag{2.4}$$

Rearranging the right-hand-side of (2.4) yields

$$b_{(j+2)} - \sum_{k=j+2}^{m} \frac{\alpha_k}{\alpha_j}(b_{(k)} - b_{(k+1)}).$$

Since $\alpha_j > \alpha_{j+1}$, it holds that

$$p_j^{VCG}(\mathbf{b}) \geq b_{(j+2)} - \sum_{k=j+2}^{m} \frac{\alpha_k}{\alpha_{j+1}}(b_{(k)} - b_{(k+1)}) = p_{j+1}^{VCG}(\mathbf{b}). \tag{2.5}$$

Therefore, $p_j^{VCG}(\mathbf{b}) = p_{j+1}^{VCG}(\mathbf{b})$ if and only if $b_{(j+1)} = b_{(j+2)} = \ldots = b_{(m+1)}$. Otherwise, it holds that $p_j^{VCG}(\mathbf{b}) > p_{j+1}^{VCG}(\mathbf{b})$. $\square$

Finally, let us introduce an additional notational convention that we will use in subsequent chapters whenever it facilitates our work. Sometimes it may be more convenient to describe the payment schemes indexed by players and not by positions. We therefore denote

$$q_i(\mathbf{b}) = p_{s(\mathbf{b},i)}(\mathbf{b})$$

for every player $i \in N$. Accordingly, let

$$\mathbf{q}(\mathbf{b}) = (q_1(\mathbf{b}), q_2(\mathbf{b}), \ldots, q_n(\mathbf{b}))$$

be the player payment scheme. The correspondence $\mathbf{p} \to \mathbf{q}$ is one-to-one. All assumptions about the position payment schemes can be transformed to analogous assumptions about the player payment schemes.

2.4 Tie-breaking rules

The most commonly used tie-breaking rule in practice is the *Rule of First-Arrival.* If two or more players submit equal bids, they are ordered according to the point in time their bids were recorded at the search engine. The player who submitted first gets the highest position among those players, the player who submitted second is assigned to the next position and so on. The *Rule of First-Arrival* is typically modeled assuming that the auctioneer uses a *random priority rule.* Formally, let Γ be the set of all permutations $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ of the set of players $N = \{1, \ldots, n\}$. The auctioneer chooses an arbitrary permutation $\gamma \in \Gamma$ with equal probability, but does not tell the players the resulting priority rule before they submit their bids. If he tells the players in advance, the tie-breaking rule is called a *fixed priority rule.* Given there are equal bids $b_i = b_k$, player i has a higher priority than player k if and only if $\gamma_i < \gamma_k$. Every vector of bids $\mathbf{b} \in \mathbf{B}$ and a permutation $\gamma \in \Gamma$ uniquely determine an allocation s. In what follows, we will assume a fixed priority rule, and without loss of generality, let the permutation applied be $\overline{\gamma} = (1, 2, \ldots, n)$, the natural order of players.

3 Analysis of Equilibria

3.1 Complete Information

The generalized second-price auction was modeled as a static one-shot game of complete information e.g. by (Edelman, Ostrovsky, Schwarz [2007]) and (Varian [2007]). Advertisers bidding on Google and Yahoo! can adjust their bids on specific keywords repeatedly. This offers sufficient space for experimentation in order to learn everything necessary to conceive other bidders' primarily private values. Furthermore, since players can adjust their bids at any point in time, stable bids in an infinitely repeated game must be static best responses to each other since otherwise bidders would have an incentive to change their bids.

In order to analyze equilibrium behavior in a static GSP auction with complete information, (Edelman, Ostrovsky, Schwarz [2007]) develop the concept of *locally envy-free equilibria*, while (Varian [2007]) analyzes boundaries of equilibrium bids in order to determine equilibrium properties. Both authors independently observe a strong relationship between position auctions and two-sided matching models and use well-known results from such assignment games in order to strengthen their findings: The generalized second-price position auctions possesses multiple equilibria, while among the set of these equilibria, the most beneficial equilibrium for the bidders and the worst for the search engine is the one that generates the same allocation s and vector of payments $\mathbf{p}^{VCG} = (p_1^{VCG}, p_2^{VCG}, \ldots, p_k^{VCG})$ as if the auctions had been designed according to the rules of Vickrey-Clarke-Groves. Formally, we assume in this section that all bidders' values are common knowledge, i.e. the vector $\mathbf{v} = (v_1, \ldots, v_n)$ is known to all players

$i \in N$. Let the valuation of a player assigned to position j be denoted by $v_{(j)}$. Let the payment of each agent assigned to position $j \in K$ be defined according to (2.1), i.e. $p_j^{GSP}(\mathbf{b}) = b_{(j+1)}$, while $p_j = 0$ for all $j \in \{m^+, -1\}$. Ties will be broken according to $\overline{\gamma}$. Let $\mathbb{G}^{C,GSP} = \{\mathbb{E}_P, \mathbf{v}, \mathbf{p}^{GSP}, \overline{\gamma}\}$ be a generalized second-price position auction under complete information. In equilibrium, each agent should prefer his current position over all other positions:

Definition 2 *(Varian [2007]) A symmetric Nash equilibrium set of prices of the game $\mathbb{G}^{C,GSP}$ satisfies*

$$\alpha_j(v_{(j)} - p_j) \geq \alpha_k(v_{(j)} - p_k) \; \forall \, j, k \in K, \tag{3.1}$$

where $p_j = b_{(j+1)}$.

In what follows, we will show an explicit characterization of equilibrium prices and bids in a series of simple arguments following (Varian [2007]). We then show the close relationship between position auctions and the assignment game first studied by (Shapley, Shubik [1972]) applying the definition of locally envy-free equilibria and implicitly prove that the two definitions yield the same outcome of equilibrium bids and allocation.

Firstly, we will examine some properties that must hold in a symmetric Nash equilibrium.

Observation 1 *A symmetric Nash equilibrium of $\mathbb{G}^{C,GSP}$ is an efficient allocation.*

Proof. Rearranging the inequalities in (3.1) gives

$$v_{(j)}(\alpha_j - \alpha_k) \geq \alpha_j p_j - \alpha_k p_k \, \forall \, j, k. \tag{3.2}$$

Now imagine two players assigned to positions $j, k \in K$. For the player in position j it must hold that

$$v_{(j)}(\alpha_j - \alpha_k) \geq p_j \alpha_j - p_k \alpha_k$$

and for the player in position k

$$v_{(k)}(\alpha_k - \alpha_j) \geq p_k \alpha_k - p_j \alpha_j.$$

Adding up these inequalities yields

$$(v_{(j)} - v_{(k)})(\alpha_j - \alpha_k) \geq 0.$$

Hence, $v_{(j)}$ and $\alpha_{(j)}$ must be ordered the same way and therefore $v_{(j-1)} \geq v_{(j)}$. Thus a symmetric Nash equilibrium is an efficient allocation. □

Observation 2 *In a symmetric Nash equilibrium of the game $\mathbb{G}^{C,GSP}$, the surplus of each player is non-negative, i.e.*

$$\alpha_j(v_{(j)} - p_j) \geq 0.$$

Proof. All players assigned to a position $j \in \{m^+, -1\}$ are not assigned to a real position and pay zero. Since $\alpha_j = 0$ for all $j \in \{m^+, -1\}$, their profit will be zero. For all players assigned to a position $j \in K$, we get from the inequalities defining a symmetric Nash equilibrium,

$$\alpha_j(v_{(j)} - p_j) \geq \alpha_{m+1}(v_{(j)} - p_{m+1}) = 0$$

since by definition $\alpha_{m+1} = 0$. □

Observation 3 *In a symmetric Nash equilibrium of the game $\mathbb{G}^{C,GSP}$, prices are monotone, i.e. $\alpha_{j-1}p_{j-1} \geq \alpha_j p_j$ for all $j \in K \cup \{m^+, -1\}$. If $v_{(j)} > p_j$ it holds that $p_{j-1} > p_j$.*

Proof. Rearranging (3.1) yields

$$p_{j-1}\alpha_{j-1} \geq p_j\alpha_j + v_{(j)}(\alpha_{j-1} - \alpha_j) \geq p_j\alpha_j.$$

Observe that

$$p_{j-1}\alpha_{j-1} \geq p_j\alpha_j + v_{(j)}(\alpha_{j-1} - \alpha_j) \geq p_j\alpha_j + p_j(\alpha_{j-1} - \alpha_j) = p_j\alpha_{j-1}.$$

Now if $v_{(j)} > p_j$, the second inequality is strict and therefore $p_{j-1} > p_j$. □

In order to show that a set of bids is a symmetric Nash equilibrium, it is sufficient to show that a player cannot gain by moving one position up or down. If it is not profitable for him to move either one slot up or down, then neither is it profitable for him to move more than one position up or down.

Observation 4 *Given a set of bids that satisfies the inequalities in (3.1) for $k = j+1$ and $k = j-1$, then it satisfies these inequalities for all $k \in K \cup \{m^+, -1\}$.*

Proof. Suppose the symmetric Nash equilibrium inequalities hold for positions $j-1$ and j and for positions j and $j+1$. It can be easily shown that it must hold for $j-1$ and $j+1$. From (3.2)—nobody wants to move down one position—we get

$$v_{(j-1)}(\alpha_{j-1} - \alpha_j) \geq p_{j-1}\alpha_{j-1} - p_j\alpha_j \tag{3.3}$$

and

$$v_{(j)}(\alpha_j - \alpha_{j+1}) \geq p_j\alpha_j - p_{j+1}\alpha_{j+1} \tag{3.4}$$

respectively. Since values are monotone, i.e. $v_{(j-1)} \geq v_{(j)}$, we get from (3.4)

$$v_{(j-1)}(\alpha_j - \alpha_{j+1}) \geq p_j\alpha_j - p_{j+1}\alpha_{j+1}. \tag{3.5}$$

Adding up (3.3) and (3.5) yields

$$v_{(j-1)}(\alpha_{j-1} - \alpha_{j+1}) \geq p_{j-1}\alpha_{j-1} - p_{j+1}\alpha_{j+1},$$

i.e. nobody wants to move down more than one position. The other direction (nobody wants to move up) works similarly. Consider the equation that states that the player in position j does not want to move up to position $j-1$,

$$\alpha_j(v_{(j)} - p_j) \geq \alpha_{j-1}(v_{(j)} - p_{j-1}) \tag{3.6}$$

and the equation stating that the player in position $j+1$ does not want to move up to position j

$$\alpha_{j+1}(v_{(j+1)} - p_{j+1}) \geq \alpha_j(v_{(j+1)} - p_j). \tag{3.7}$$

Note that (3.6) remains valid after replacing $v_{(j)}$ by $v_{(j+1)}$. Adding up the modified equations (3.6') and (3.7) yields

$$\alpha_{j+1}(v_{(j+1)} - p_{j+1}) \geq \alpha_{j-1}(v_{(j+1)} - p_{j-1}). \tag{3.8}$$

This concludes the proof. □

With these observations in hand, we are able to explicitly characterize equilibrium bids and prizes of the game $\mathbb{G}^{C,GSP}$. Recall that in a symmetric Nash equilibrium, a player assigned to position j does not want to move down one position, i.e.

$$\alpha_j(v_{(j)} - p_j) \geq \alpha_{j+1}(v_{(j)} - p_{j+1})$$

as well as a player assigned to position $j+1$ does not want to move up one position, i.e.

$$\alpha_{j+1}(v_{(j+1)} - p_{j+1}) \geq \alpha_j(v_{(j+1)} - p_j).$$

Putting these two inequalities together yields

$$v_{(j)}(\alpha_j - \alpha_{j+1}) + p_{j+1}\alpha_{j+1} \geq p_j\alpha_j \geq v_{(j+1)}(\alpha_j - \alpha_{j+1}) + p_{j+1}\alpha_{j+1}. \tag{3.9}$$

Since $p_j = b_{(j+1)}$, (3.9) can be rewritten as

$$v_{(j-1)}(\alpha_{j-1} - \alpha_j) + b_{(j+1)}\alpha_j \geq b_{(j)}\alpha_{j-1} \geq v_{(j)}(\alpha_{j-1} - \alpha_j) + b_{(j+1)}\alpha_j. \tag{3.10}$$

Let $\mu_j = \frac{\alpha_j}{\alpha_{j-1}} < 1$. Equation (3.10) then becomes

$$v_{(j-1)}(1 - \mu_j) + b_{(j+1)}\mu_j \geq b_{(j)} \geq v_{(j)}(1 - \mu_j) + b_{(j+1)}\mu_j. \tag{3.11}$$

Thus, the bid $b_{(j)}$ of a player assigned to position j is bounded above and below by a convex combination of the bid of the player below him and, on the upper bound, of the value of the player assigned to the position above him or, at the lower bound, of his own value. Now any recursively chosen sequence of bids that satisfies the equivalent inequalities (3.9)-(3.11) is a symmetric pure strategy Nash equilibrium of the game $\mathbb{G}^{C,GSP}$. If we write down the solutions to these recursions of the upper and lower bounds,

$$b^U_{(j)} = \frac{v_{(j-1)}(\alpha_{j-1} - \alpha_j) + b_{(j+1)}\alpha_j}{\alpha_{j-1}},$$

$$b^L_{(j)} = \frac{v_{(j)}(\alpha_{j-1} - \alpha_j) + b_{(j+1)}\alpha_j}{\alpha_{j-1}},$$

we get, starting with the fact that there are only m positions and $\alpha_{m+1} = 0$,

$$b^U_{(j)} = \frac{\sum_{k=j}^{m+1} v_{(k-1)}(\alpha_{k-1} - \alpha_k)}{\alpha_{j-1}}, \tag{3.12}$$

$$b^L_{(j)} = \frac{\sum_{k=j}^{m+1} v_{(k)}(\alpha_{k-1} - \alpha_k)}{\alpha_{j-1}}. \tag{3.13}$$

Now notice that since

$$p_j^{GSP}(\mathbf{b}^L) = b^L_{(j+1)} = \frac{\sum_{k=j+1}^{m+1} v_{(k)}(\alpha_{k-1} - \alpha_k)}{\alpha_j},$$

the payment of each player assigned to position j at the lower bound symmetric Nash equilibrium of the generalized second-price auction coincides with the payment in the corresponding game $\mathbb{G}^{VCG} = \{\mathbb{E}_P, \mathbf{v}, \mathbf{p}^{VCG}, \overline{\gamma}\}$, i.e. a position auction with the same environment $\mathbb{E}$ and tie-breaking rule $\overline{\gamma}$, but payment rule designed according to the rules of Vickrey-Clarke-Groves. Since values and prices are monotone in the generalized second-price auction as well as under VCG, the allocation $\mathbf{s}$ is also the same and therefore the outcome of the lower bound symmetric Nash equilibrium of $\mathbb{G}^{GSP}$ is equivalent to the equilibrium outcome in $\mathbb{G}^{VCG}$. The revenue for the search engine becomes $\sum_{j=1}^{m+1} b^{GSP}_{(j)}$. Since $b^L_{(j)}$ are the lowest symmetric Nash equilibrium bids, the equilibrium in which all bidders bid at the lower bound is the one that generates the lowest profit among all symmetric Nash equilibria for the search engine, and vice versa, this equilibrium is the most profitable for the bidders. Given this scenario, the revenue for the search engine becomes $\sum_{j=1}^{m+1} b^{GSP,L}_{(j)} = \sum_{j=1}^{m} p_j^{VCG}$. $\square$

Another interesting line of argumentation is to point out the strong relationship between position auctions and the assignment game first studied by (Shapley, Shubik [1972]). For that purpose, we will present another equilibrium definition in the game $\mathbb{G}^{C,GSP}$ introduced by (Edelman, Ostrovsky, Schwarz [2007]):

Definition 3 *A vector of bids* $\mathbf{b}$ *in the game* $\mathbb{G}^{C,GSP}$ *is a locally envy-free equilibrium if a player cannot improve his payoff by exchanging bids with the player ranked one position above him. More formally, in a locally envy-free equilibrium, for every player* $i \in N$*, we have* $\alpha_{s(\mathbf{b},i)}(v_i - p_{s(\mathbf{b},i)}) \geq \alpha_{s(\mathbf{b},i)-1}(v_i - p_{s(\mathbf{b},i)-1})$.

For the matter of notational convenience, it may be more useful to define equilibrium requirements indexed by positions. Then, for every position $j \leq \min\{N, K\}$, we have that in a locally envy-free equilibrium, $\alpha_j(v_{(j)} - p_j) \geq \alpha_{j-1}(v_{(j)} - p_{j-1})$. Note that this requirement is analogous to the requirement stated in the definition above.

OBSERVATION 5 *Every locally envy-free equilibrium is a symmetric Nash equilibrium.*

Proof. By definition, in any locally envy-free equilibrium, no player can gain from being rematched with the player assigned to the position directly above him. He also cannot profitably rematch with a player assigned to a position below him since this would contradict the assumption of being in equilibrium: If such a profitable rematching existed, he could slightly underbid the respective player and get his position and payment. It is also not profitable for a player in a locally envy-free equilibrium to rematch with a player assigned to a position $k < j - 1$. This follows immediately from second part of the proof of Observation 4. Therefore, given the restrictions of a locally envy-free equilibrium, for any player assigned to a position $j \in K \cup \{m^+, -1\}$ it holds that

$$\alpha_j(v_{(j)} - p_j) \geq \alpha_k(v_{(j)} - p_k) \; \forall \, j, k \in K \cup \{m^+, -1\}.$$

But this is exactly the definition of a symmetric Nash equilibrium of the game $\mathbb{G}^{C,GSP}$.
□

In order to see the strong relationship between the position auction problem and the assignment game studied by (SHAPLEY, SHUBIK [1972]), we may view each position as an agent who is looking for the most profitable match with an advertiser. Furthermore, only one advertiser can be matched with a position respectively. The utility of an advertiser i who is matched with position j is given by $u_i(j) = \alpha_j v_i$. The advertiser pays p_j. His payoff becomes $\omega_i(j) = \alpha_j v_i - p_j$. Let us denote the assignment game by A. The assignment problem is to solve for the matching of advertisers with positions that maximizes, in our case, advertisers joint payoff. It can be shown that the problem can be solved without the need of a central authority, which would just assign the position with the highest click-through rate to the advertiser with the highest valuation, the position with the second highest click-through rate to the advertiser with the second highest valuation and so on, by means of a price mechanism, such that

$$u_i(j) - p_j \geq u_i(k) - p_k \; \forall \, j, k. \tag{3.14}$$

Thus, given the prices p_j for all positions $j \in K$, advertiser i would prefer the position he holds to any other position. Now since $u_i(j) = \alpha_j v_i$ and $p_j = \alpha_j b_{(j+1)}$, (3.14) becomes

$$\alpha_j(v_{(j)} - p_j) \geq \alpha_k(v_{(j)} - p_k) \,\forall\, j, k.$$

However, this is exactly the definition of the symmetric Nash equilibrium inequalities of the game $\mathbb{G}^{C,GSP}$. Therefore, any symmetric Nash equilibrium of $\mathbb{G}^{C,GSP}$ is just a competitive equilibrium of the corresponding assignment game A.
We can also show the analogy explicitly:

LEMMA 3 *(*ASHLAGI ET AL. *[2008]) The outcome of any locally envy-free equilibrium of the game $\mathbb{G}^{C,GSP}$ is a stable assignment.*

Proof. This follows immediately from the proof of Observation (5). Since all players prefer the position they hold to any other position, the assignment must be stable. ☐

LEMMA 4 *(*ASHLAGI ET AL. *[2008]) If the number of bidders is greater than the number of available positions, then any stable assignment is an outcome of a locally envy-free equilibrium of the game $\mathbb{G}^{C,GSP}$.*

Proof. By a result of (SHAPLEY, SHUBIK [1972]), a stable assignment must be efficient and assortative. Let the corresponding prices in the stable assignment be denoted by p_j^A. It is easy to construct an outcome of the game $\mathbb{G}^{C,GSP}$ that corresponds to the outcome of a stable assignment of the corresponding assignment game A. Therefore, let $b_{(1)} = \max v_i$ and $b_{(j)} = p_{j-1}^A \,\forall\, j > 1$. Since $p_j^{GSP} = b_{(j+1)}$, the payments of the position auction correspond to those of the stable assignment. Furthermore, since the stable assignment is efficient, it must hold that $p_j > p_{j+1}$ for all j. Therefore it holds that $b_{(j)} > b_{(j+1)}$. Hence, the allocation $\mathbf{s}$ is also the same in the original stable assignment as well as in the corresponding position auction. Now we only need to show that the vector of bids $\mathbf{b}$ constitutes a locally envy-free equilibrium. Since no player can profitably move to another position in the originally stable assignment, and being matched to another position in game $\mathbb{G}^{C,GSP}$ yields exactly the same profit for the respective player as if he would have been assigned to the respective position in the originally stable assignment of the game A, it cannot be profitable for him in $\mathbb{G}^{C,GSP}$. $\square$

Given Lemma 3 and 4, we find another way to prove that the VCG equivalent outcome of $\mathbb{G}^{C,GSP}$ is the best outcome for the bidders and the worst for the search engine by finding an analogous result for the assignment game A. Recall the vector of payments $\mathbf{p}^{VCG} = (p_1^{VCG}, p_2^{VCG}, \dots, p_k^{VCG})$ such that

$$p_j^{VCG}(\mathbf{b}) = \frac{\sum_{k=j+1}^{m+1} b_{(k)}(\alpha_{k-1} - \alpha_k)}{\alpha_j} \;\; \forall\, j \in K \cap \forall\, \mathbf{b} \in \mathbf{B}$$

We already proved that this vector of payments constitutes a symmetric Nash equilibrium of the game $\mathbb{G}^{C,GSP}$. Now consider any stable assignment of the game A. It must hold that $p_m \geq v_{(m+1)}$ since otherwise the bidder assigned to position $m+1$ would find it profitable to match with position m. But it also holds that $p_m^{VCG} = b_{(m+1)}$ and since VCG is truthful, $p_m^{VCG} = b_{(m+1)} = v_{(m+1)}$. Hence in the buyer optimal stable assignment, $p_m = p_m^{VCG}$. Next, in any stable assignment, it must hold that

$$(\alpha_{m-1} - \alpha_m)p_{m-1} + (p_{m-1} - p_m)\alpha_m = \alpha_{m-1}p_{m-1} - \alpha_m p_m \geq v_{(m)}(\alpha_{m-1} - \alpha_m)$$

since otherwise the player assigned to position m would find it profitable to move up to position $m-1$. Hence we get

$$p_{m-1} \geq \frac{v_{(m)}(\alpha_{m-1} - \alpha_m) + \alpha_m p_m}{\alpha_{m-1}} \geq \frac{v_{(m)}(\alpha_{m-1} - \alpha_m) + \alpha_m v_{(m+1)}}{\alpha_{m-1}} = p_{m-1}^{VCG}.$$

Therefore, in the buyer-optimal stable assignment, $p_{m-1} = p_{m-1}^{VCG}$. Now if we proceed by induction, we get $p_j = p_j^{VCG}$ for any $j \in K$ in the buyer optimal stable assignment, and hence, the profit for the search engine aggregates to $\sum_{j=1}^{m} p_j^{VCG}$. Obviously, this is the worst symmetric Nash equilibrium from the perspective of the search engine. $\square$

3.2 Incomplete Information

(Lahaie [2006]) analyzed position auctions under complete and incomplete information. The findings in the complete information case are similar to those already presented.

For the case of incomplete information, explicit formulas of equilibrium bids under alternating first-price payment rules are derived. This works out to be very similar to the first-price equilibrium in the single-item model, see for example (Krishna [2002]), with the difference that equilibrium bids depend on the density of the second highest value among all N bidders in the single-item case, while it depends on a weighted combination of the densities for the second, third, etc. highest values in a position auction. Unfortunately, as a result from attempting to derive explicit equilibrium bids given a second-price payment rule, (Lahaie [2006]) states that *"the resulting differential equations for this case do not have a neat analytical solution."* Nevertheless, (Ashlagi et al. [2008]) as well as (Edelman, Ostrovsky, Schwarz [2007]) find an ex post equilibrium of the generalized second-price auction, though both slightly change the model in order to do so. The former use a mediator to implement the equilibrium, while the latter develop an analogue of the standard English auction, the so-called *generalized English auction* in order to answer the question of how bidders converge to the equilibrium found under complete information. This ex post equilibrium is exactly the bidder optimal one in which players get VCG payoffs. Let us briefly present the ideas of (Edelman, Ostrovsky, Schwarz [2007]) before we introduce the model of (Ashlagi et al. [2008]) in more detail since, it will find extensive use in subsequent chapters.

So, how do bidders converge to a situation in which nobody has an incentive to change bids anymore? Actually, it can be shown that there are simple strategies that do the job: each bidder starts bidding at zero and increases his bid as long as his payoff increases as well. In order to model this procedure, imagine a clock showing the current price that continuously increases over time. The bid of a player is then determined by the price the clock shows at the moment he drops out. The game is over when all bidders but one have dropped out, and the price each bidder pays is equal to the price at which the bidder directly before him dropped out. The players are assigned positions according to the point in time they dropped out, with the last remaining player being assigned to the top position. Bidders' values need not be a common value. The resulting game of incomplete information has a unique perfect Bayesian equilibrium, which, as already mentioned, coincides with the VCG equivalent equilibrium of $\mathbb{G}^{C,GSP}$. The equilibrium is ex post, i.e. the equilibrium strategies of the players are best responses to other bidders' strategies independent of the values actually realized.

In order to model position auctions under incomplete information, some notational preliminaries are necessary. In what follows, we will model position auctions as pre-Bayesian games following (Ashlagi et al. [2008]). Pre-Bayesian games are games of incomplete information where the probability measure over the profiles of types of player is not common knowledge. (This is in contrast to the model used by (Edelman, Ostrovsky, Schwarz [2007]). The common solution concept of pre-Bayesian games is the search for ex post equilibria. Formally, let there be a fixed number of players $N = \{1, \ldots, n\}$. Each players' set of actions is given by $b_i \in B_i$. For each game, there is a set of outcomes $\mathbf{O}$. Every action profile, that is any vector of actions $\mathbf{b} \in \mathbf{B} = B_1 \times B_2 \times \ldots \times B_n$ yields an outcome $a \in \mathbf{O}$. The function that maps action profiles to outcomes is denoted by $\psi : \mathbf{B} \to \mathbf{O}$. In general, there is a set of states $\theta \in \Theta$. In pre-Bayesian games, the payoff of any player i depends on the realized state $\theta \in \Theta$ as well as on the outcome $a \in \mathbf{O}$, i.e. $\omega_i(\theta, a)$. Since we deal with private information only, we assume $\Theta = \mathbf{V} = V_1 \times \ldots \times V_n$. To keep things simple, we will assume that the payoff will only depend on the type v_i and not on the vector of types $\mathbf{v} = (v_1, \ldots, v_n)$. Therefore, a pre-Bayesian game is given by $\mathbb{G}^B = (N, \mathbf{V}, \mathbf{O}, (\omega_i)_{i \in N}, \mathbf{B}, \psi)$. Finally, let the utility of player i, $u_i : V_i \times \mathbf{B} \to \mathbb{R}$ be defined as:

$$u_i(v_i, \mathbf{b}) = \omega_i(v_i, \psi(\mathbf{b})).$$

Given the general definition of pre-Bayesian games, we will see that position auctions fit very well into the concept of pre-Bayesian games, a fact that leaves us with a model of position auctions that deals with incomplete information: Let $V_i = [0, \infty)$ be the set of types of i and let $\mathbf{V} = V_1 \times V_2 \times \ldots \times V_n$ be the set of states. Let $B_i = [0, \infty)$ be the set of actions of i and $\mathbf{B} = B_1 \times B_2 \times \ldots \times B_n$ be the set of action profiles. Let $\mathbf{S}$ be the set of allocations. An outcome of a position auction is an allocation s, i.e. an assignment of players to positions, and a vector of players' payments $\mathbf{q}$. (In order to avoid entanglement, recall that the player payment scheme can be written analogously indexed by positions, and the one-to-one correspondence $\mathbf{p} \to \mathbf{q}$ of the different payment schemes. We will use either of both notations dependent on which is more convenient respectively.) Let $\mathbf{O} = \mathbf{S} \times \mathbb{R}_+^n$ be the set of outcomes. The function $\psi : \mathbf{B} \to \mathbf{O}$ that maps actions profiles to outcomes becomes

$$\psi(\mathbf{b}) = (s(\mathbf{b}), \mathbf{q}(\mathbf{b})).$$

The payoff for each bidder, $\omega_i : V_i \times \mathbf{O} \to \mathbb{R}$ is then defined as

$$\omega_i(v_i, (\mathbf{s}, \mathbf{q})) = \omega_i(v_i, (s_1, \ldots, s_n, q_1, \ldots, q_n)) = \alpha_{s_i}(v_i - q_i).$$

The utility of player i, $u_i : V_i \times \mathbf{B} \to \mathbb{R}$ becomes

$$u_i(v_i, \mathbf{b}) = \alpha_{s(\mathbf{b},i)}(v_i - q_i(\mathbf{b})) = \alpha_{s(\mathbf{b},i)}(v_i - p_{s(\mathbf{b},i)}).$$

Since there is in fact no difference in the definition of players' payoffs to the definition in the very basic case introduced in Chapter 2, let the generalized second-price auction under incomplete information that we are going to analyze be given by $\mathbb{G}^{B,GSP} = \{\mathbb{E}_P, \mathbf{V}, \mathbf{O}, \mathbf{B}, \psi, \overline{\gamma}\}$, where $\psi(\mathbf{b}) = (s(\mathbf{b}), \mathbf{q}^{GSP}(\mathbf{b}))$. In $\mathbb{G}^B$, the strategy of player i is a function f_i that assigns an action $b_i = f(v_i) \in B_i$ to every possible type $v_i \in V_i$. Now we are able to define an ex post equilibrium:

Definition 4 *An ex post equilibrium in $\mathbb{G}^B$ is a profile of strategies $\mathbf{f} = (f_1, \ldots, f_n)$ such that for every player $i \in N$ and for every $\mathbf{v} \in \mathbf{V}$ it holds that*

$$u_i(v_i, \mathbf{f}(\mathbf{v})) \geq u_i(v_i, b_i, \mathbf{f_{-i}}(\mathbf{v_{-i}})), \quad \forall\, b_i \in \mathbf{B}_i$$

where $\mathbf{f}(\mathbf{v}) = (f_1(v_1), \ldots, f_n(v_n))$ and $\mathbf{f_{-i}}(\mathbf{v_{-i}})$ is the vector of players strategies without i. f_i is a weakly dominant strategy for i if the above inequalities hold for all $\mathbf{f_{-i}}(\mathbf{v_{-i}})$.

The reader may excuse the elaborate effort in notational preliminaries, but it actually gives us a clean and comprehensive structure of position auctions and the instruments to introduce and clearly point out the role of mediators in the given framework. With these observations at hand, we are able to turn towards the next chapter, in which we will introduce some theory on the concept of mediators and finally present a mediation device which implements the VCG equivalent outcome in a generalized second-price auction under incomplete information.

4 Mediators in Position Auctions

So far, we have learned that in position auctions bidders can profit if they are able to coordinate to the VCG equivalent equilibrium among the multiple equilibria that exist. In this chapter we address the question as to how such a coordination can be facilitated. We show that this is possible by the use of a *mediation service* or *mediation device*. For that purpose, we introduce the concept of a *mediator* and apply it to the context of position auctions. A mediator is a reliable entity that can play on behalf of the bidders, but cannot enforce behavior. Any player is free to choose either to give the mediator the right to play on his behalf or to participate in the respective game directly without the help of the mediator.

However, players must not only coordinate to the most profitable among multiple equilibria. It can be shown that, with the help of a mediator, bidders can even reach outcomes, in which the payoffs for the agents are beyond the convex hull of Nash equilibrium payoffs. Let us illustrate the power of such a mediator by a simple example which goes back to (Monderer, Tennenholtz [2006]). Therefore, consider the classical prisoner's dilemma game:

	Cooperate	Defect
Cooperate	4,4	0,6
Defect	6,0	1,1

It is a well-known fact that in the unique Nash equilibrium of the game both agents choose to defect, since this is a dominant strategy for both. In equilibrium, each player ends up with a payoff of 1. But this equilibrium is inefficient. If they could coordinate to the outcome (C, C), i.e. both choose to cooperate, each player would get a payoff of 4. Now consider a mediator that comes into play and announces that he will choose 'Cooperate' on behalf of the players if and only if *both* players give him the right to play on their behalf. If only one player gives him the right to play, he will choose 'Defect' on his behalf. The new game generated by the mediator is in the form:

	Mediator	Cooperate	Defect
Mediator	4,4	6,0	1,1
Cooperate	0,6	4,4	0,6
Defect	1,1	6,0	1,1

In this game, it is a weakly dominant strategy for each player to use the mediators service and hence, both players end up with a payoff of 4. In fact, the mediator chooses 'Cooperate' on behalf of the players and therefore successfully implements the outcome (C, C) of the original game, which is beyond the convex hull of Nash equilibrium payoffs. In Chapter 5, we will develop a collusive mediator that coordinates the bidders in the position auction game $\mathbb{G}^{C,GSP}$ to a collusive outcome that is beyond the convex hull of Nash equilibrium payoffs.

The critical point is the mediator's reliability. Due to his reliability, the mediator is able to coordinate players on outcomes of a game in which, originally, they would have had an individual incentive to deviate given that the other players acting accordingly, and therefore such an outcome cannot be realized without the use of a mediator. The underlying simple idea is that the mediator announces, and even more importantly, actually implements the advantageous outcome if and only if all players give him the right to play. If not all players give him the right to play, he acts on behalf of the subset of players that give him the right to play in a way that a less profitable outcome for all players is obtained. In a position auction, this is very easily done by punishing the players who do not give him the right to play by using very high bids by the players that use his service. But this is not a very sophisticated choice since it might discourage players to use the service of the mediator since they may end up with negative payoffs.

In equilibrium, they would not end up with a negative payoff, but if players believe that errors can occur with positive probability, they might be distracted. At least in practice this seems to us a great obstacle. Therefore we will focus on *individually rational* mediators. An individually rational mediator can guarantee each player who uses his service and reports his type truthfully a nonnegative payoff—no matter how the players who do not use the service of the mediator play and independent of the reports sent to the mediator by the other players who decided to give him the right to play on their behalf—even if they report false values.

One may argue that the mediator's reliability is a critical point itself, i.e. how his reliability is enforced. (Ashlagi et al. [2008]) argue that the mediator simply maximizes his own profit and implicitly assume that this is highly correlated with the goal of guaranteeing high utility to rational players. However, note that a mediator may be just a software protocol that players can observe in advance. This seems to be a more convincing argument to us.

By his existence, the mediator creates a new game, the *mediated game* $\mathbb{G}^{\mathbf{m}}$, but without changing the fundamental rules of the original game $\mathbb{G}$. Given the new game $\mathbb{G}^{\mathbf{m}}$, it is ideally an ex post equilibrium for all players to give the mediator the right to play and, in a game of incomplete information, report their types truthfully, while he implements the more profitable outcome for the players in the original game $\mathbb{G}$.

In order to reach a desired outcome for the participants of a certain game, one might as well chose a mechanism design approach, i.e. change the rules of the game such that the desired outcome will be in equilibrium. However, this might be problematic since it would require the modification of existing standards. In face of the vast amounts of money that Google and Yahoo! earn with online advertising, this would impose a considerable economic risk. Therefore, the use of a mediator has to be a superior choice in many settings.

We will now introduce the concept of a mediator more formally and present some advanced theory. We will then present a mediator that implements the VCG equivalent outcome in a generalized second-price auction under incomplete information.

4.1 Model

When introducing the concept of a mediator, we need to distinguish between games of complete and incomplete information, i.e. pre-Bayesian games in our case. In a game of complete information, the mediator simply asks for the "right to play" on behalf of the players and performs actions $\mathbf{b} = (\mathbf{b}_S)_{S\subseteq N}$ on behalf of the subset of players S that give him the right to play. In a pre-Bayesian game, the mediator does not know the valuations of the players and needs to be informed by the players about their types. Let us define a mediator for pre-Bayesian games following (Ashlagi et al. [2008]) as follows:

Definition 5 *Let $\mathbb{G}^B = (N, \mathbf{V}, \mathbf{O}, (\omega_i)_{i\in N}, \mathbf{B}, \psi)$ be a pre-Bayesian game. A mediator for $\mathbb{G}^B$ is a vector $\mathbf{m} = (\mathbf{m}_s)_{S\subseteq N}$, where $\mathbf{m}_s : \mathbf{V}_S \to \mathbf{B}_S$.*

Let the set of players who give the mediator the right to play be S. Given the members of S send the mediator their profile of types $\mathbf{v}_S = (v_i)_{i\in S} \in \mathbf{V}_S$, the mediator performs actions $\mathbf{m}_S(\mathbf{v}_S) \in \mathbf{B}_S$ on their behalf. The set of actions of any player i in the new game created by the existence of the mediator therefore dilates to $B_i^{\mathbf{m}} = B_i \cup V_i$. Since a player can either send a message of his type to the mediator or participate in the game directly, we assume that $B_i \cap V_i = \emptyset$. Choosing $b_i^m = b_i \in B_i$ means that i is not using the service of the mediator, while choosing $b_i^m = v_i \in V_i$ means that the player reports the type v_i to the mediator and at the same time grants the mediator the permission to play on his behalf. Let $b^{\mathbf{m}} \in \mathbf{B}^{\mathbf{m}}$. We denote by $N(\mathbf{b}^{\mathbf{m}}) = \{i \in N : b_i^{\mathbf{m}} \in V_i\}$ the set of players that use the service of the mediator. Let $-N(\mathbf{b}^{\mathbf{m}}) = N \backslash N(\mathbf{b}^{\mathbf{m}})$. By the existence of a mediator we obtain a new function that maps actions to outcomes. Let us denote the new function by $\psi^{\mathbf{m}} : \mathbf{B}^{\mathbf{m}} \to \mathbf{O}$. It is defined as follows:

$$\psi^{\mathbf{m}}(\mathbf{b}^{\mathbf{m}}) = \psi(\mathbf{m}_{N(b^{\mathbf{m}})}(\mathbf{b}^{\mathbf{m}}_{N(\mathbf{b}^{\mathbf{m}})}), \mathbf{b}^{\mathbf{m}}_{-N(\mathbf{b}^{m})})$$

The *mediated game* is therefore given by $\mathbb{G}^{\mathbf{m}} = (N, \mathbf{V}, \mathbf{O}, (\omega_i)_{i\in N}, \mathbf{B}^{\mathbf{m}}, \psi^{\mathbf{m}})$. Let the utility of player i be denoted by $u_i^{\mathbf{m}}$. Note that the mediated game differs from the original game only in the set of actions of players and the function that maps actions to outcomes. All other components remain unaltered. Now we are able to define a *mediated equilibrium*, i.e. an equilibrium in the mediated game as follows:

DEFINITION 6 *(*ASHLAGI ET AL. *[2008]) Let $\mathbb{G}^B$ be a pre-Bayesian game, and let $\varphi : \mathbf{V} \to \mathbf{O}$ be an outcome function. We say that φ is a mediated equilibrium in $\mathbb{G}^B$ if there exists a mediator for $\mathbb{G}^B$, $\mathbf{m}$, and an ex post equilibrium $\mathbf{g} = (g_1, \ldots, g_n)$ in $\mathbb{G}^{\mathbf{m}}$, such that $g_i(v_i) \in \mathbf{V}_i$ for every $i \in N$ and for every $v_i \in V_i$, and it holds that*

$$\varphi(\mathbf{v}) = \psi(\mathbf{m}_N(\mathbf{g}(\mathbf{v}))) \quad \forall\, \mathbf{v} \in \mathbf{V}.$$

Let us call the strategy of player i in the mediated game, which is, give the mediator the right to play and report the type truthfully, the $T-$ strategy. Accordingly, let us call the strategy profile in which all players report truthfully the $T-$ strategy profile. Let $\varphi^{\mathbf{m}} : \mathbf{V} \to \mathbf{O}$ denote the outcome function generated by the mediator when every player is using the $T-$ strategy, i.e.

$$\varphi^{\mathbf{m}}(\mathbf{v}) = \psi(\mathbf{m}_N(\mathbf{v})) \quad \forall\, \mathbf{v} \in \mathbf{V}.$$

We say that the mediator *truthfully implements* $\varphi^{\mathbf{m}}$ in $\mathbb{G}^B$ if the T-strategy profile is an ex post equilibrium in $\mathbb{G}^{\mathbf{m}}$, that is $g_i(v_i) = v_i$ for every player $i \in N$. Finally, note that the well-known *revelation principle* that can be applied, again following (ASHLAGI ET AL. [2008]):

OBSERVATION 6 *(*ASHLAGI ET AL. *[2008]) Let $\mathbb{G}^B$ be a pre-Bayesian game, and let $\varphi : \mathbf{V} \to \mathbf{O}$ be an outcome function. φ is a mediated equilibrium if and only if there exists a mediator $\mathbf{m}$ that implements φ by truthful mediation.*

Now let us apply these definitions to the context of position auctions. Observe that our meanwhile familiar generalized second-price auction is a pre-Bayesian game $\mathbb{G}^{B,GSP} = \{\mathbb{E}_P, \mathbf{V}, \mathbf{O}, \mathbf{B}, \psi, \overline{\gamma}\}$ with $\psi(\mathbf{b}) = (s(\mathbf{b}), \mathbf{q}^{GSP}(\mathbf{b}))$. Now each bidder can either participate in the auction directly, i.e. submit a bid $b_i \in B_i$, or report a type $\hat{v}_i$ to the mediator, which must not necessarily be his true type v_i. The action set of each player is therefore $B_i \cup V_i$. If $S = N(\mathbf{b}^{\mathbf{m}})$ is the set of players that send a type to the mediator and therefore give him the right to play, the mediator bids $\mathbf{m}_S(\hat{\mathbf{v}}_S)$ on their behalf. With these observations at hand, let us define an ex post equilibrium of the mediated game $\mathbb{G}^{B,GSP,\mathbf{m}}$.

Definition 7 *(*Ashlagi et al. *[2008]) The $T-$strategy profile is an ex post equilibrium in the mediated game $\mathbb{G}^{B,GSP,\mathbf{m}}$ if for every player i and type v_i, and for every vector of types of the other players, $\mathbf{v}_{-i}$, the following two conditions hold:*

1. *Any player i is not better off when he gives the mediator the right to play and reports a false type. That is, for every $\hat{v}_i \in V_i$*

$$u_i(v_i, \mathbf{m}_N(v_i, \mathbf{v}_{-i})) \geq u_i(v_i, \mathbf{m}_N(\hat{v}_i, \mathbf{v}_{-i})).$$

2. *Any player i is not better off when he bids directly. That is, for every $b_i \in B_i$,*

$$u_i(v_i, \mathbf{m}_N(v_i, \mathbf{v}_{-i})) \geq u_i(v_i, b_i, \mathbf{m}_{N\setminus\{i\}}(\mathbf{v}_{-i})).$$

In any ex post equilibrium in which players report truthfully, the mediator implements an outcome function in the original game $\mathbb{G}^{B,GSP}$. Let us denote this outcome function by $\varphi^{\mathbf{m}} : \mathbf{V} \to \mathbf{O}$, which is defined as follows:

$$\varphi^{\mathbf{m}}(\mathbf{v}) = \psi(\mathbf{m}_N(\mathbf{v})) = (s(\mathbf{m}_N(\mathbf{v})), q(\mathbf{m}_N(\mathbf{v})))$$

Finally, recall that we are searching for a mediator that can guarantee each player who decides to use his service and reports his type truthfully, a nonnegative payoff—independent of whatever action profile is chosen by the other players. Formally, let us define an *individually rational mediator* as follows:

Definition 8 *A mediator $\mathbf{m}$ in any pre-Bayesian game $\mathbb{G}^B$ is said to be individually rational if it holds that for every $S \subseteq N$ and every player $i \in S$, the level of utility is nonnegative, i.e.*

$$u_i(v_i, \mathbf{m}_S(\mathbf{v}_S, \mathbf{b}_{-S})) \geq 0 \quad \forall\, \mathbf{b}_{-S} \in \mathbf{B}_{-S} \;\; \textit{and} \quad \forall\, \mathbf{v}_S \in \mathbf{V}_S.$$

Prepared with these theoretical preliminaries, we are finally able to construct a mediator that will implement the VCG equivalent outcome in a generalized second-price auction under incomplete information.

4.2 A Mediator for the Generalized Second-Price Auction

We need to find a mediator such that it is an ex post equilibrium in the mediated game to report valuations truthfully to the mediator for every player $i \in N$. This is due to the revelation principle as we have already stated in Observation 6. The mediator then implements the same allocation and the same vector of payments for every player as if the auction were designed according to the rules of VCG. Let us denote this outcome function by $\varphi^{VCG} : \mathbf{V} \to \mathbf{O}$ and let it be defined as follows:

$$\varphi^{VCG}(\mathbf{v}) = (s(\mathbf{v}), \mathbf{q}^{VCG}(\mathbf{v})).$$

Formally, let us define a mediator that implements the VCG outcome function in $\mathbb{G}^{B,GSP}$ as follows:

Definition 9 *Let* $\mathbf{m}$ *be a mediator for* $\mathbb{G}^{B,GSP}$*. We say that* $\mathbf{m}$ *implements the VCG outcome function in* $\mathbb{G}^{B,GSP}$*, i.e.* φ^{VCG}*, if the* $T-$ *strategy profile is an ex post equilibrium in the mediated game* $\mathbb{G}^{B,GSP,\mathbf{m}}$*, and it holds that* $\varphi^{\mathbf{m}} = \varphi^{VCG}$*.*

Let us now present such an individually rational mediator in more detail. The mediator and the proofs in this chapter are once again due to (Ashlagi et al. [2008]). Finally, the following mediator implements φ^{VCG} in $\mathbb{G}^{B,GSP}$:

Proposition 2 *The VCG Mediator.*

(a) For every $\mathbf{v} \in V$ *let* $\mathbf{m}_N(\mathbf{v}) = \mathbf{b}(\mathbf{v})$*, where* $\mathbf{b}(\mathbf{v})$ *defined as follows:*

- $\mathbf{b}_i(\mathbf{v}) = p^{VCG}_{s(\mathbf{v},i)-1}(\mathbf{v})$ *for every player* i *such that* $2 \leq s(\mathbf{v}, i) \leq m + 1$*.*
- $\mathbf{b}_i(\mathbf{v}) = \frac{\mathbf{b}_i(\mathbf{v})}{1+\rho}$ *for every player* i *such that* $s(\mathbf{v}, i) > m + 1$ *and for some arbitrary but fixed* $\rho > 0$*.*
- $\mathbf{b}_i(\mathbf{v}) = \varepsilon + p^{VCG}_1(\mathbf{v})$ *for the player* i *such that* $s(\mathbf{v}, i) = 1$ *and for some arbitrary but fixed* $\varepsilon > 0$*.*

(b) For every strict subset $S \subset N$, $\mathbf{m}_S(\mathbf{v}_S) = \mathbf{v}_S$ for every $\mathbf{v}_S \in \mathbf{V}_S$.

Proof. In order to prove that the VCG mediator effectively implements φ^{VCG}, we have to show that the resulting allocation s is consistent with the vector of valuations $\mathbf{v}$, the resulting vector of payments is equivalent to the vector of payments that would occur under VCG, and that it is an ex post equilibrium for every player to give the mediator the right to play and report his type truthfully.

First, assume every player is using the $T-$ strategy. Let us once again denote by $v_{(i)}$ the ith highest among all valuations. Assume that the vector of valuations is generic, there are no two valuations that are equal: $v_{(1)} > \ldots > v_{(m)} > \ldots > v_{(n)}$. Therefore, in a VCG position auction, a player with the ith highest valuation will be assigned to the ith highest position and pays p_i^{VCG}. Note that for all players assigned to a position $k > m$ it holds that $p_i^{VCG} = 0$ since they are not assigned to a real position and therefore do not have to pay anything. Now if the players report their types truthfully to the mediator, he submits the vector of bids $\mathbf{b}(\mathbf{v}) = (b_{(1)}(\mathbf{v}), \ldots, b_{(n)}(\mathbf{v}))$ to the generalized second-price auction, where

$$\mathbf{b}(\mathbf{v}) = (\varepsilon + p_1^{VCG}(\mathbf{v}), p_1^{VCG}(\mathbf{v}), p_2^{VCG}(\mathbf{v}), \ldots, p_m^{VCG}(\mathbf{v}), \frac{p_m^{VCG}(\mathbf{v})}{1+\rho}, \ldots, \frac{p_m^{VCG}(\mathbf{v})}{1+\rho}).$$

From Lemma 2, we know that, since $v_{(1)} > \ldots > v_{(m)} > \ldots > v_{(n)}$, $p_1^{VCG}(\mathbf{v}) > p_2^{VCG}(\mathbf{v}) > \ldots > p_m^{VCG}(\mathbf{v})$. Hence, it holds that $b_{(1)}(\mathbf{v}) > b_{(2)}(\mathbf{v}) > \ldots > b_{(m+1)}(\mathbf{v}) > b_{(i)}(\mathbf{v})$ for every $i > m+1$. Therefore, the vector of bids $\mathbf{b}(\mathbf{v})$ submitted to the generalized second-price auction generates the same allocation s as the vector $\mathbf{v}$ submitted to the VCG position auction. Furthermore, since in the generalized second-price auction, $p_j^{GSP}(\mathbf{b}) = b_{(j+1)}$, it holds that every player assigned to a position $i \in K$ pays $p_i^{VCG}(\mathbf{v})$. Hence, $\varphi^{\mathbf{m}}(\mathbf{v}) = \varphi^{VCG}(\mathbf{v})$.

However, we have to prove that this result remains true when there are ties in the vector $\mathbf{v}$. We have to show that $\varphi^{\mathbf{m}}(\mathbf{v}) = \varphi^{VCG}(\mathbf{v})$ for an arbitrary vector of valuations $\mathbf{v} \in \mathbf{V}$. The proof in this case is a bit more involving. Nevertheless, let us start with showing that $s(\mathbf{b}(\mathbf{v})) = s(\mathbf{v})$ for every vector of valuations $\mathbf{v} \in \mathbf{V}$. Therefore, we consider a pair of players $i \neq l$. We have to show that whenever $1 \leq s(\mathbf{v}, i) < s(\mathbf{v}, l) \leq m+1$, $s(\mathbf{b}(\mathbf{v}), i) < s(\mathbf{b}(\mathbf{v}), l)$. Consider the following three cases:

1. $s(\mathbf{v}, i) = 1$.

   Therefore, the bid submitted by the mediator on behalf of player i is $\mathbf{b}_i(\mathbf{v}) = \varepsilon + p_1^{VCG}(\mathbf{v})$. But for player l, it must hold that $b_l(\mathbf{v}) \leq \max_{k=1}^{n} p_k^{VCG}(\mathbf{v})$. Hence, $b_i(\mathbf{v}) > b_l(\mathbf{v})$ and thus $s(\mathbf{b}(\mathbf{v}), i) < s(\mathbf{b}(\mathbf{v}), l)$.

2. $s(\mathbf{v}, i) > 1$, $s(\mathbf{v}, l) > m + 1$.

   The bid submitted by the mediator on behalf of player l is therefore $b_l(\mathbf{v}) = \frac{p_m^{VCG}(\mathbf{v})}{1+\rho} < p_m^{VCG}(\mathbf{v})$. On the contrary, player i is assigned to a real slot and by the second part of Lemma 2, it must hold that $b_i(\mathbf{v}) \geq p_m^{VCG}(\mathbf{v})$. Therefore, $b_l(\mathbf{v}) < b_i(\mathbf{v})$ and thus $s(\mathbf{b}(\mathbf{v}), i) < s(\mathbf{b}(\mathbf{v}), l)$.

3. $s(\mathbf{v}, i) > 1$ and $s(\mathbf{v}, l) \leq m + 1$. Consider two different cases:

   – $v_i = v_l$. Since we assume a fixed priority rule and the permutation applied is by assumption the natural order of players $\overline{\gamma}$, i has a higher priority than l, and therefore $s(\mathbf{v}, i) < s(\mathbf{v}, l)$. Hence, by the second part of Lemma 2, $p_{s(\mathbf{v},i)-1}^{VCG}(\mathbf{v}) \geq p_{s(\mathbf{v},l)-1}^{VCG}(\mathbf{v})$. Thus, $b_i(\mathbf{v}) \geq b_l(\mathbf{v})$ and hence it must hold that $s(\mathbf{b}(\mathbf{v}), i) < s(\mathbf{b}(\mathbf{v}), l)$.

   – $v_i > v_l$. Once again, from the second part of Lemma 2, we know that $p_{s(\mathbf{v},i)-1}^{VCG}(\mathbf{v}) > p_{s(\mathbf{v},i)}^{VCG}(\mathbf{v})$. The inequality is strict since $b_{s(\mathbf{v},i)} = v_i > b_{s(\mathbf{v},l)} = v_l$. Now since $s(\mathbf{v}, i) < s(\mathbf{v}, l) - 1$, we know that $p_{s(\mathbf{v},i)}^{VCG}(\mathbf{v}) \geq p_{s(\mathbf{v},l)-1}^{VCG}(\mathbf{v})$. Thus $p_{s(\mathbf{v},i)-1}^{VCG}(\mathbf{v}) \geq p_{s(\mathbf{v},l)-1}^{VCG}(\mathbf{v})$ and therefore $b_i(\mathbf{v}) > b_l(\mathbf{v})$. Consequently, $s(\mathbf{b}(\mathbf{v}), i) < s(\mathbf{b}(\mathbf{v}), l)$.

Thus we showed that $s(\mathbf{b}(\mathbf{v})) = s(\mathbf{v})$ for every $\mathbf{v} \in \mathbf{V}$. Now, according to the rules of the generalized second-price auction, for every player i who is assigned to a position $s(\mathbf{v}, i) \in K$, it holds that $p_{s(\mathbf{b}(\mathbf{v}),i)}(\mathbf{b}(\mathbf{v})) = b_{(j+1)} = p_{s(\mathbf{v},i)}^{VCG}(\mathbf{v})$. All players who are assigned to a dummy position $j \geq m + 1$ and therefore do not receive a real spot pay zero. Hence, $q_i(\mathbf{b}(\mathbf{v})) = q_i^{VCG}(\mathbf{v})$ for every player $i \in N$. Thus, $\mathbf{q}(\mathbf{b}(\mathbf{v})) = \mathbf{q}^{VCG}(\mathbf{v})$ for all $\mathbf{v} \in \mathbf{V}$ and consequently, $\varphi^{\mathbf{m}} = \varphi^{VCG}$.

Finally, we have to show that the $T-$ strategy profile is an ex post equilibrium in $\mathbb{G}^{B,GSP,\mathbf{m}}$. Let $\mathbf{v} \in \mathbf{V}$ be an arbitrary vector of valuations and assume that every player but i uses the $T-$ strategy. Obviously, it is not beneficial for i to report a false value to the mediator since in a VCG position auction it is a weakly dominant strategy for every player to report his valuation truthfully as we have already shown in Lemma 1. Now the only strategy that remains to player i beside reporting truthfully to the mediator is to participate in the auction directly. Therefore, let the vector of valuations reported to the mediator by the other players be denoted by $\mathbf{v}_{-i}$, and let b_i be the bid of player i. Suppose that, if player i had reported truthfully to the mediator, he would have been assigned to position $k = s(\mathbf{v}, i)$. Since $\varphi^{\mathbf{m}} = \varphi^{VCG}$, it holds that $s(\mathbf{b}(\mathbf{v}), i) = k$. Now consider player i who is assigned to a position $j = s((\mathbf{v}_{-i}, b_i), i)$ if he deviates and bids b_i. If $j = k$, his profit will not be affected, since his payment does not depend on his own bid, and therefore, the deviation is not profitable. Now if $j \notin K$, his profit will be zero, and therefore it is not profitable for i. Finally, assume that $j \in K$. Recall that if player i does not give the mediator the right to play, the mediator bids $\mathbf{m}_{N\setminus\{i\}}(\mathbf{v}) = \mathbf{v}_{-i}$ on behalf of the players $N\setminus\{i\}$. Therefore, the vector submitted to the generalized-second-price auction becomes $\tilde{\mathbf{b}} = (\mathbf{v}_{-i}, b_i)$. It holds that

$$\alpha_k(v_i - p_k(\mathbf{b}(\mathbf{v}))) = \alpha_k(v_i - p_k^{VCG}(\mathbf{v})) \geq$$

$$\alpha_j(v_i - p_j^{VCG}(\tilde{\mathbf{b}})) \geq \alpha_j(v_i - \tilde{b}_{(j+1)}).$$

The first inequality is due to the truthfulness of VCG, whereas the second inequality follows from the first part of Lemma 2. But $\tilde{b}_{(j+1)}$ is exactly the payment that player i has to make in the position auction if he deviates, and therefore it holds that

$$\alpha_k(v_i - p_k(\mathbf{b}(\mathbf{v}))) \geq \alpha_j(v_i - p_j(\tilde{\mathbf{b}})).$$

Thus, participating in the auction directly is not a profitable strategy for player i. This concludes the proof. □

OBSERVATION 7 *The VCG mediator is individually rational.*

Proof. If all players use the $T-$ strategy, the outcome in the generalized second-price auction is exactly the same outcome as generated in a VCG position auction with the vector of valuations $\mathbf{v}$. From Lemma 2 we know that for any position $j \in K$, $p_j^{VCG}(\mathbf{v}) \leq b_{(j+1)}$, and therefore for every player i with $s(\mathbf{v}, i) \in K$ it holds that

$$p_{s(\mathbf{v},i)}^{VCG}(\mathbf{v}) \leq b_{(s(\mathbf{v},i)+1)} = v_{(s(\mathbf{v},i)+1)} \leq v_i.$$

All players who are not assigned to a real slot pay zero anyway and thus cannot end up with a negative payoff. If only a subset of players use the service of the mediator, the mediator bids $\mathbf{m}_S(\mathbf{v}) = \mathbf{v}_S$ on behalf of those players, and by the rules of the generalized second-price auction, none of those players will pay more than his or her value. $\square$

5 Collusion in a Generalized Second-Price Auction

But is there a possibility for the bidders to reach an even more profitable outcome than VCG? Can they do so by using a mediator? Do other forms of cartel exist that exploit the search engine? Again, the information structure is crucial in order to answer these questions. In the introductory example on mediators, we presented a mediator that coordinated players in a game of complete information on the collusive outcome which was beyond the convex hull of Nash equilibrium payoffs. But is this also possible in a generalized second-price auction? If so, does the result hold under incomplete information? These questions will be subject of the forthcoming sections. We will present a mediator that does the job under complete information. In the latter case, bidders can effectively act as a single agent and maximize their joint payoff. However, under incomplete information, it is impossible to implement any kind of collusive outcome by means of a mediator. This is due to the fact that every anonymous truth-revealing position auction is necessarily a Vickrey-Clarke-Groves position auction. Furthermore, we will give a short review on a well-studied form of cartel in single-item auctions, so-called *pre-auction knockouts*, and show that such a cartel does not work for GSP. Finally, we will investigate how the situation changes if we allow for repeated play.

5.1 A Collusive Mediator

With complete information, bidders can effectively act as a single agent and maximize their joint payoff. All the search engine can do in response to colluding bidders is to set an optimal reserve price. Should this be the case, the mechanism design problem would be reduced to simple textbook monopoly pricing. We will not model the task of finding an optimal reserve price for the search engine, but we want to show how bidders can easily exploit the search engine by the use of a mediator. Recall that under complete information, all the mediator can do is to collect the right to play on behalf of the bidders and perform actions $\mathbf{b} = (\mathbf{b}_S)_{S\subseteq N}$ on behalf of the subset of players S that give him the right to play. Given a fixed priority rule, let, without loss of generality, the permutation applied to be once again $\overline{\gamma}$. Denote by $T_j,\ j = 1, \ldots, J$ subsets of players whose valuations, ordered by their magnitude, are consistent with the natural order $\overline{\gamma}$. For every two players $i, j \in T_j$ it holds that $(v_i - v_j)(i - j) \leq 0$. Furthermore, let us order the subsets of players T_j such that $\min\{v_i \mid i \in T_j\} > \max\{v_i \mid i \in T_{j+1}\}$. Therefore, T_1 denotes the subset of players with the highest valuations among all players, T_J denotes the subset of players with the lowest valuations among all players. Note that T_j can be singleton. Without loss of generality, let us assume that the search engine sets a positive reserve price $r > 0$. Finally, let us denote by T_c the subset of players which contains the *critical* player i_c for which it holds that either $s(\mathbf{v}, i_c) = m$, i.e. he is the player with the lowest valuation among those who are assigned to a real position, or, $v_{i_c} \geq r > v_{(i_c+1)}$, i.e. he is the player with the lowest valuation that is still greater or equal to the reserve price. Let us denote by κ the smallest increment, and let us assume that κ is infinitesimally small. The following mediator exploits the search engine on behalf of the bidders in a bidder-optimal way, and in equilibrium, all players choose to use the service of the mediator:

Proposition 3 *The Collusive Mediator.*

(a) *If all bidders decide to use the service of the mediator, let* $\mathbf{m}_N(\mathbf{v}) = \mathbf{b}(\mathbf{v})$*, where* $\mathbf{b}(\mathbf{v})$ *is defined as follows:*

- $b_i(\mathbf{v}) = r + (c - j) \cdot \kappa$ *for every player* $i \in T_j < T_c$
- $b_i(\mathbf{v}) = r$ *for every player* $i \in T_c$ *such that* $v_i \geq v_{i_c}$
- $b_i(\mathbf{v}) = 0$ *for every player* $i \in T_c$ *such that* $v_i < v_{i_c}$ *and for every player* $i \in T_j > T_c$

(b) For every strict subset $S \subset N$, $\mathbf{m}_S(\mathbf{v}_S) = \mathbf{v}_S$ *for every* $\mathbf{v}_S \in \mathbf{V_S}$.

Proof. Observe that the vector of bids $\mathbf{m}_N(\mathbf{v}) = \mathbf{b}(\mathbf{v})$ as defined above is by construction the smallest possible vector of bids that yields an efficient allocation. Let us denote the collusive outcome generated by the mediator by $\varphi^{\mathbf{m}^C}$. In order to show that it is an equilibrium strategy in the mediated game $\mathbb{G}^{C,GSP,\mathbf{m}}$ for every player $i \in N$ to use the service of the mediator, assume all players but one decide to use his service. First, assume that $s(\mathbf{v}, i) > m$, i.e. $v_i < v_{i_c}$. If player i decides to use the service of the mediator, he would not be assigned to a real position and hence end up with a profit of zero. Now if he deviates and bids an amount of b_i directly in the auction, the mediator submits the vector of bids $\mathbf{m}_{-i}(\mathbf{v}_{-i}) = \mathbf{v}_{-i}$ on behalf of the other players $N \backslash \{i\}$. Therefore, if $v_{i_c} \geq b_i \geq v_i$, he will not be assigned to a real position and therefore this deviation is not profitable. If $b_i \geq v_{i_c} \geq v_i$, player i will have to pay an amount that is greater or equal to his valuation and therefore end up with a profit of zero in the best case. Hence, bidding directly in the auction is not profitable for him. Now assume $s(\mathbf{v}, i) \leq m$. If player i deviates and participates directly in the auction, once again the vector submitted by the mediator on behalf of the other players $N \backslash \{i\}$ becomes $\mathbf{m}_{-i}(\mathbf{v}_{-i}) = \mathbf{v}_{-i}$. Therefore, player i's payment, if he is assigned to a position $j \in K$, will be $p_j = b_{(j+1)} = v_{(j+1)}$. If $i \in T_c$ and i is assigned to a real position j, observe that it holds that $p_j \geq v_{i_c} \geq r$, while if he uses the service of the mediator, his payment will be r. If $i \in T_j < T_c$, observe that his payment for position j will be $p_j = v_{(j+1)} \geq r + (c - j) \cdot \kappa$ since κ is the smallest increment and assumed to be infinitesimally small. Therefore, the payment of player i in any outcome of the mediated game $\mathbb{G}^{C,GSP,\mathbf{m}}$ is at least as high as in the outcome generated by the mediator, $\varphi^{\mathbf{m}^C}$, if all players use its service. Thus, it cannot be profitable for him to bid in the auction directly. $\square$

Observation 8 *The collusive mediator is individually rational.*

Proof. Observe that if not all players participate in the auction by using of the mediator, he submits the vector $\mathbf{m}_S(\mathbf{v}_S) = \mathbf{v}_S$ for the subset of players S that give him the right to play. By the rules of the auction, no player $i \in S$ will pay more than his bid, and thus, every player that gives the mediator the right to play will end up with a profit that is nonnegative. Now assume all players to give the mediator the right to play. For any player $i \in T_c$ who is assigned to a real position it holds that v_i is at least as high as r, the price he has to pay in $\varphi^{\mathbf{m}}$. For any player $i \in T_j < T_c$, the payment will be $p_j = r + (c - j) \cdot \kappa$. Now since $\min\{v_i \mid i \in T_j\} > \max\{v_i \mid i \in T_{j+1}\}$ for every $j = 1, \ldots, J$, the valuation of a player in $T_j < T_c$ is at least as high as $r + (c - j) \cdot \kappa$ since κ is the smallest possible increment. Hence, $u_i(\mathbf{m}_S(\mathbf{v}_S), \mathbf{b}_{-S}) \geq 0$ for every player $i \in S \subseteq N$. $\square$

We just showed that a collusive mediation device exists that is individually rational and exploits the search engine in a bidder optimal way. Though it makes sense to model the generalized second-price auction under complete information in order to learn about equilibrium bids and prices, it is not very useful if we investigate its vulnerability to collusion. Bidders will need a long period of experimentation in order to learn about the vector of valuations $\mathbf{v}$ and may face losses within this period. Now if a new bidder enters the market, this process must start again to the extent that incumbent bidders have to learn about the valuation of the entrant and modify the mediation device respectively. Therefore, it would be eligible to develop a mediator that implements $\varphi^{\mathbf{m}^C}$ in a game of incomplete information by truthful mediation. Unfortunately, such a mediator does not exist, as we will show in the next chapter.

5.2 The Case of Incomplete Information

In this section we will show that no mediator exists that implements an outcome that is more profitable for the bidders than φ^{VCG} by truthful mediation and at the same time preserves an efficient allocation of bidders to positions. Implicitly, this means that the collusive mediator introduced above fails to implement $\varphi^{\mathbf{m}^C}$ under incomplete information. We will show that every anonymous truth-revealing position auction is necessarily a Vickrey-Clarke-Groves position auction. But if the mediator can enforce truth-telling by the bidders only by mimicking VCG, he is restricted at best to the VCG equivalent outcome. Furthermore, since the revelation principle applies to our setting as stated in Observation 6, we do not lose any outcome by restricting attention on outcomes that can be implemented by truthful mediation. Let us summarize these findings as follows:

THEOREM 1 *Efficiency implies noncooperative profit levels—even by the use of a mediator—in any position auction $\mathbb{G}^B$.*

In the following, we will present a series of arguments in order to prove Theorem 1. The main part of the proof goes back to (ASHLAGI [2008]): We start by introducing a notion of anonymity of the allocation rule s. We then prove that every position auction with an anonymous allocation rule is necessarily a VCG auction. What is new is that we combine these results with the findings we already presented on mediators, which completes the proof.

Therefore, let us present some findings on truth-revealing position auctions and introduce some additional notational preliminaries before we get involved with the actual proof. Let us call an allocation rule s *implementable* if there exist a payment scheme $\mathbf{p}$ such that in the resulting position auction it is a weakly dominant strategy for every player i to report his utility truthfully. Furthermore, we say that an allocation rule s is *monotone* if and only if it holds that

$$(b_i - b_i')(\alpha_{s_i(b_i,\mathbf{b}_{-i})} - \alpha_{s_i(b_i',\mathbf{b}_{-i})}) \geq 0$$

for every two bids $b_i, b_i' \in V_i$ and for every fixed $\mathbf{b}_{-i}$. The click-through rate of any bidder i cannot decrease given he raises his bid and the bids of the players $N\backslash\{i\}$ are not changed. Equivalently, given the bids of the other players $N\backslash\{i\}$ to be fixed, it cannot happen that player i will be assigned to a lower position as a result of raising his bid. The following lemma is due to (BIKHCHANDANI ET AL. [2006]):

Lemma 5 *For position auctions it holds that an allocation rule is implementable if and only if it is monotone.*

Proof. The authors show that weak monotonicity is a sufficient condition of dominant-strategy incentive compatibility for deterministic social choice functions in a model with multidimensional types, i.e. $V_i \in \mathbb{R}^m_+$ for any $m \geq 1$, private values and quasi-linear preferences. Our model fits well into this setting with $m = 1$. $\square$

Furthermore, let us apply a result obtained by (Holmstrom [1979]) to the context of position auctions since it will be useful in the following:

Lemma 6 *(*Holmstrom *[1979]) Let s be a welfare maximizer. If the allocation rule s is implementable by a payment scheme $\mathbf{q}$, $(s, \mathbf{q})$ is a VCG position auction.*

Proof. The author showed that the allocation rule in a truth-revealing mechanism determines the payment function of each agent in a payment scheme up to an additive constant if the set of valuations of every player is convex. Since in our model every V_i is convex, Lemma 6 emerges to hold. $\square$

Until now, we have only dealt with standard VCG position auctions and in fact, we have already shown explicitly, that in a standard VCG position auction, the resulting allocation rule s is a welfare maximizer since $s_i(\mathbf{v}) < s_j(\mathbf{v})$ if and only if $v_i \geq v_j$. Furthermore, we explicitly showed that s is implementable and observed monotonicity. But a variety of VCG position auctions exist. Let us introduce the notion of non-standard VCG position auctions. Non-standard VCG position auctions share the same allocation rule with standard ones, but payments for each player i differ from standard payments by an additional payment that does only depend on the bids of the other players $N \backslash \{i\}$. A non-standard VCG position auction would be an intuitive choice in order to enforce a collusive protocol by a mediation device since they inherit the most desirable property from standard VCG position auctions, i.e. revealing valuations truthfully. Unfortunately, they finally fail to establish a cartel for various reasons, as we will show in the following. Let us formally define a VCG position auction by $(s, \hat{\mathbf{q}})$, where s is the resulting allocation rule of a standard VCG position auction and $\hat{\mathbf{q}}$ is a vector of payments for which it holds that for every player i and for every vector of bids $\mathbf{b} \in \mathbf{V}$,

$$\hat{q}_i(\mathbf{b}) = q_i(\mathbf{b}) + g_i(\mathbf{b}_{-i}), \tag{5.1}$$

where $q_i(\mathbf{b})$ is the standard VCG payment function of player i and $g_i : \mathbf{V}_{-i} \to \mathbb{R}$. Before we state an important finding, we need to introduce the notion of *seller rationality.* An incentive compatible position auction $(s, \mathbf{q})$ is *seller rational* if for every player i and every profile of bids $\mathbf{b} \in \mathbf{V}$ it holds that $q_i(\mathbf{b}) \geq 0$. Recall the notion of individual rationality. From Lemma 2 we know that every standard VCG position auction satisfies both seller and individual rationality. We will see that this is not true for non-standard VCG position auctions, as stated in the following lemma:

LEMMA 7 *(*ASHLAGI *[2008]) Every individually rational and seller rational VCG position auction is a standard VCG position auction.*

Proof. Let $(s, \hat{\mathbf{q}})$ be a VCG position auction as defined above. In order to prove that every individually and seller rational VCG position auction is a standard one, we have to show that for every player i and for every vector of bids $\mathbf{b}_{-i}$, $g_i(\mathbf{b}_{-i}) = 0$. Therefore, suppose in negation that a player i and a function $g_i(\mathbf{b}_{-i}) \neq 0$ exist for every $\mathbf{b}_{-i} \in \mathbf{V}_{-i}$. Assume first that $g_i(\mathbf{b}_{-i}) < 0$. But this implies that for every $v_i < |g_i(\mathbf{b}_{-i})|$, since (by the first part of Lemma 2) $p_j(\mathbf{b}) < b_{(j+1)} < b_{(j)}$, $\hat{q}_{(i)}(v_i, \mathbf{b}_{-i}) \leq v_{(i+1)} + g_{(i)}(\mathbf{b}_{-i}) \leq v_{(i)} + g_{(i)}(\mathbf{b}_{-i}) < 0$ — which contradicts seller rationality. Now suppose in negation that $g_i(\mathbf{b}_{-i}) > 0$. We will construct a vector of bids such that we contradict individual rationality. Therefore, let $b_i = \inf\{x_i \in V_i | s_i(x_i, \mathbf{b}_{-i}) \in K\}$. Monotonicity implies that $s_i(b_i + \varepsilon, \mathbf{b}_{-i}) < s_i(b_i, \mathbf{b}_{-i})$. Now since in a VCG position auction it holds that $p_m(\mathbf{b}) = b_{(m+1)}$ and by the second part of Lemma 2, it must hold that for a player with a valuation of $b_m + \varepsilon$ it holds that $q_i(b_m + \varepsilon, \mathbf{b}_{-i}) \geq b_m$. Now for every $0 < \varepsilon < g_i(\mathbf{b}_{-i})$ this implies that $\hat{q}_i(b_i + \varepsilon, \mathbf{b}_{-i}) = q_i(b_i + \varepsilon, \mathbf{b}_{-i}) + g_i(\mathbf{b}_{-i}) \geq b_i + \varepsilon$. Hence, a player with valuation $v_i = b_i + \varepsilon$ would pay more than his valuation — a contradiction of individual rationality. $\square$

Finally, some more notational preliminaries are needed. For every profile of bids $\mathbf{b} \in \mathbf{V}$, let us denote by $\mathbf{b}^{ij}$ the bid profile that results from $\mathbf{b}$ if the valuations of players i, j are exchanged, i.e.

$$\mathbf{b}^{ij} = (b_1, \ldots, b_{j-1}, b_i, b_{j+1}, \ldots, b_{i-1}, b_j, b_{i+1}, \ldots, b_n).$$

Recall that we call b_i to be distinct in $\mathbf{b}$ if $b_i \neq b_j$ for every player $j \neq i$. Furthermore, let us call the vector of bids $\mathbf{b}$ to be generic if b_i is distinct for every player $i \in N$. We are now able to introduce a notion of anonymity:

Definition 10 *(*Ashlagi *[2008]) An allocation rule s is anonymous if the following holds: For every player i, and for every $\mathbf{b} \in \mathbf{V}$ such that b_i is distinct in $\mathbf{b}$, $s_j(\mathbf{b}^{ij}) = s_i(\mathbf{b})$ for every player j.*

Observe that this states that if the bid of player i is distinct in $\mathbf{b}$ and player i exchanges bids with player j, it must hold that j receives the previous position of i, but i must not necessarily be assigned to the previous position of j due to possible ties. If the bid of player j is also distinct in $\mathbf{b}$, player i receives the former position of j.

Proposition 4 *(*Ashlagi *[2008]) A truth-revealing position auction with an anonymous allocation rule is a VCG position auction.*

Proof. We need some more notational preliminaries. Let $\tilde{\mathbf{V}} = \{\mathbf{b} \in \mathbf{V} : b_1 = b_2 = \ldots = b_n\}$. Let $H(\mathbf{b}) = \{i \in N | b_i = b_{(1)}\}$ be the set of players that submitted the highest bid. Furthermore, let $h(\mathbf{b})$ be the the position with the highest index in $H(\mathbf{b})$, i.e. the worst position a highest bidder is allocated. Formally, $h(\mathbf{b})$ is defined as follows:

$$h(\mathbf{b}) = \max\{k \in K | \; \exists \; i \in H(\mathbf{b}) : s_i(\mathbf{b}) = k\}.$$

Similarly, let $v(\mathbf{b})$ be the position with the lowest index among all bidders with a non-highest bid, i.e.

$$v(\mathbf{b}) = \min\{k \in K | \; \exists \; b_{(k)} < b_{(1)}\}.$$

Accordingly, the set of bidders that submitted a bid equal to the one of the respective player in position $v(\mathbf{b})$ is denoted by $V(\mathbf{b}) = \{i \in N | b_i = b_{(v(\mathbf{b}))}\}$. With these notational preliminaries at hand, we are able to start with the actual prove. Therefore, let $(s, \mathbf{p})$ be a truth-revealing position auction with an anonymous allocation rule s. From Holmstrom's lemma, we know that, in order to show that $(s, \mathbf{p})$ is a VCG position auction, we only need to prove that s is a welfare maximizer, i.e. s is consistent with the vector of valuations $\mathbf{v}$, or equivalently, since we have a truth-revealing position auction with the vector of bids $\mathbf{b}$. Now the only thing we need to show is that a highest bidder can not be assigned to a lower position than a non-highest bidder, i.e. $v(\mathbf{b}) \geq h(\mathbf{b})$ for every vector of bids $\mathbf{b} \in \mathbf{B}$. By a recursive argument, this implies that the allocation s is preserved with the order of bids. We simply denote for every $\mathbf{b}_{-H(\mathbf{b})}$, $v(\mathbf{b}_{-H(\mathbf{b})}) \geq h(\mathbf{b}_{-H(\mathbf{b})})$ and the proof follows. For every vector of bids $\mathbf{b} \in \tilde{\mathbf{V}}$, any allocation rule s is a welfare maximizer since all participants in the auction are highest bidders. For every $\mathbf{b} \in \mathbf{V} \backslash \tilde{\mathbf{V}}$, the proof is more elaborate, which is why we only want to sketch the idea and refer to (Ashlagi [2008]) for the details. He provides a proof by contradiction. He supposes in negation that a bid profile $\mathbf{b}$ exists such that $v(\mathbf{b}) < h(\mathbf{b})$. Initially assuming that $|V(\mathbf{b})| = 1$, (Ashlagi [2008]) gets by a series of manipulations of the vector of bids and by extensive use of monotonicity of s, that every highest bidder and i must be in the first $v(\mathbf{b})$ positions, contradicting that $|H(\mathbf{b}) \cup \{i\}| \geq v(\mathbf{b}) + 1$. But this must be the case if $v(\mathbf{b}) < h(\mathbf{b})$. In the case that $|V(\mathbf{b})| > 1$, he just derives another vector of bids $\mathbf{b}^l$ with the same properties than the initial vector of bids $\mathbf{b}$ until it holds that $|V(\mathbf{b}^l)| = 1$, which leaves us with the same contradiction as above. □

We are now able to prove the following proposition very easily, which is our key insight in order to finally prove Theorem 1:

Proposition 5 *(Ashlagi [2008]) Every truth-revealing, individually rational and seller rational position auction with an anonymous allocation rule is necessarily a standard VCG position auction.*

Proof. By Proposition 4, every truth-revealing position auction $(s, \mathbf{p})$ with an anonymous allocation rule must be a VCG position auction, and by Lemma 7, it must necessarily be a standard VCG position auction. □

By Observation 6, we know that an outcome function $\varphi : \mathbf{V} \to \mathbf{O}$ is a mediated equilibrium in a pre-Bayesian game $\mathbb{G}^{\mathbf{B}} = (N, \mathbf{V}, \mathbf{O}, (\omega_i)_{i\in N}, \mathbf{B}, \psi)$ if and only if there exists a mediator that implements φ by truthful mediation, i.e. $\varphi^{\mathbf{m}}(\mathbf{v}) = \psi(\mathbf{m}_N(\mathbf{v}))$. Let us consider the generalized second-price auction $\mathbb{G}^{B,GSP}$, which fits into the set of pre-Bayesian games. Now by Proposition 5, the only truth-revealing, individually rational and seller rational position auction is a standard VCG position auction, i.e. $\psi^{VCG} = (s, \mathbf{p}^{VCG})$ is the only outcome $a \in \mathbf{O}$ that can be implemented in a position auction $\mathbb{G}^{\mathbf{B}}$ while revealing the valuations of players truthfully. By Definition 5, a mediator in a pre-Bayesian game $\mathbb{G}^{B}$ cannot facilitate transfer payments, he can only play in the game $\mathbb{G}^{B,GSP}$ on behalf of the players and therefore create the new *mediated game* $\mathbb{G}^{B,GSP,\mathbf{m}} = (N, \mathbf{V}, \mathbf{O}, (\omega_i)_{i\in N}, \mathbf{B}^{\mathbf{m}}, \psi^{\mathbf{m}})$. Recall that this game differs from the original game only in the set of actions of players $\mathbf{B}^{\mathbf{m}}$ and the function $\psi^{\mathbf{m}}$ that maps action profiles to outcomes, i.e. $\psi^{\mathbf{m}}(\mathbf{b}^{\mathbf{m}}) = \psi(\mathbf{m}_{N(b^{\mathbf{m}})}(\mathbf{b}^{\mathbf{m}}_{N(\mathbf{b}^{\mathbf{m}})}), \mathbf{b}^{\mathbf{m}}_{-N(\mathbf{b}^{m})})$. Now since the sets of possible outcomes $\mathbf{O}$ in the two games are equivalent, and the underlying mechanism that maps action profiles to outcomes, i.e. the function $\psi : \mathbf{B} \to \mathbf{V}$ varies only by its arguments, ψ^{VCG} remains the only outcome that can be implemented in $\mathbb{G}^{B,GSP,\mathbf{m}}$ and has the property to reveal valuations truthfully. But a mediator who has to stick to the rules of $\mathbb{G}^{\mathbf{B}}$, i.e. to the function $\psi : \mathbf{B} \to \mathbf{V}$, can only ensure the players to reveal their valuations truthfully by a set of prices with the respective property, i.e. $\psi^{\mathbf{m}}(\mathbf{b}^{\mathbf{m}}) = \psi(\mathbf{m}_{N(b^{\mathbf{m}})}(\mathbf{b}^{\mathbf{m}}_{N(\mathbf{b}^{\mathbf{m}})}), \mathbf{b}^{\mathbf{m}}_{-N(\mathbf{b}^{m})}) = (s, \mathbf{p}^{VCG})$. Hence, no mediator exists that can implement an outcome $\varphi^{\mathbf{m}^C}$ that is more profitable for the bidders than $\varphi^{\mathbf{m}}(\mathbf{v}) = \psi^{VCG}(\mathbf{v})$ and preserve efficiency at the same time. This completes the proof of Theorem 1. $\square$

We just saw that no mediator exists who is able to establish an efficient cartel. We want to introduce a new form of a mediation device in the following chapter, i.e. a mediator who is equipped with the ability to facilitate transfer payments among players, and investigate if such a mediator is able to successfully establish an efficient cartel.

5.3 Collusion with Side Payments

The most basic model of collusion was introduced in a seminal paper by (Graham, Marshall [1987]). It was further extended by (McAffee, McMillan [1992]), who investigate both weak and strong cartels. A strong cartel is a consortium of players that is able to exclude new entrants and realize transfer payments among the members of the cartel. However, one of cartel's main difficulties is to find a way to divide the profits of the collusive agreement since each member has an incentive to claim for an even bigger share of the spoils. In order to do so, a common way is to conduct prior to the auction a so-called *pre-auction knockout (PAKT).* In (McAffee, McMillan [1992]), it is shown that an incentive compatible, efficient mechanism exists that is implementable by a pre-auction. The idea is fairly simple. The bidder who wins the pre-auction participates in the legitimate auction, while all other members of the cartel stay absent. The bidder then pays each of the losing bidders in the pre-auction an equal share of the difference between his bid in the knockout and the price he actually pays in the legitimate auction, which is in general equal to the reserve price if the cartel is all-inclusive. Such a mechanism can easily be thought of as a kind of mediation device. The mediator conducts the pre-auction and bids the reserve price on behalf of the winning bidder and zero on behalf of all other members of the cartel in the legitimate auction. The only difference to the mediator above is his ability to facilitate side payments.

However, the mechanism breaks down if we relax the assumption of the cartel being able to exclude new entrants. High profits earned by successful collusion attract so-called parasite bidders who only participate in the auction in order to participate in the cartel's sharing of profits—although they would never actually win the auction. If the number of parasite bidders is large enough, the share of cartel's profits for each member diminishes. Therefore, the cartel must discourage the entry of parasite bidders. This can be done by offering zero profits to bidders with low values. However, this implies that some bidders receive zero transfers and thus, any transfer scheme cannot be lump-sum. Therefore, the highest bidder that receives no transfer has an incentive to overstate his valuation. (McAffee, McMillan [1992]) summarize:

"Collusion contains the seeds of its own destruction."

Our impossibility result in the previous section is aligned with another finding of (McAffee, McMillan [1992]). For a single-item first-price auction, they state:

> *"If a cartel member whose valuation is less than or equal to the minimum price r is constrained to earn zero profits, efficiency implies noncooperative profit levels."*

A mediator that searches for a way to implement an efficient collusive scheme in the generalized second-price auction is required to align incentives in such a way that bidders do not overstate their valuations. However, this is only possible by a set of noncooperative prices if the entry of parasite bidders cannot be prevented and thus the incorporation of side payments is no longer an option. Therefore, independent of the existence of an incentive compatible and efficient mechanism for the generalized second-price auction that incorporates side payments (given new entrants can be excluded), no collusive scheme exists that exploits the search engine and preserves efficiency at the same time.

A natural next step for future work would be to investigate if a modified bid rotation scheme in the sense of Theorem 1 in (McAffee, McMillan [1992]) can be used to guarantee bidders in a generalized second-price auction higher levels of utility than those obtained by noncooperative play. Such a bid rotation scheme would instruct every bidder to submit a bid equal to the reserve price whenever his valuation exceeds r or zero otherwise, i.e.

$$b_i(v_i, \mathbf{v}_{-i}) = \begin{cases} 0 & v_i < r \\ r & v_i \geq r \end{cases} \tag{5.2}$$

Note that by the use of a mediator who bids according to 5.2 if all players give him the right to play, i.e.

$$\mathbf{m}_N(\mathbf{v}) = \begin{cases} 0 & v_i < r \\ r & v_i \geq r \end{cases} \tag{5.3}$$

and $\mathbf{m}_S(\mathbf{v}_S) = \mathbf{v}_S$ otherwise, it does not pay off to report a false value to the mediator. However, it is still necessary to prove whether bidders earn higher profits by using the service of the mediator than by bidding in the auction directly and if the resulting profits are higher than in any noncooperative equilibrium, since the mediator is useless otherwise. If the search engine employs a random priority rule, each bidder who submits a positive bid is assigned to a position $j \in K \cup \{m^+\}$ arbitrarily. If the auctioneer uses a fixed priority rule, there can be thought of any arbitrary mechanism that instructs bidders to bid similar to the mediator presented in Proposition 3, in particular as if every T_j was singleton. The bid rotation scheme obviously abandons efficiency. Nevertheless, in the case of a single-item first-price auction with free entry, a similar scheme maximizes bidders' expected profits. In particular, profits are greater than in any noncooperative equilibrium.

5.4 Repeated Play

In Section 3.2, we stated that the VCG equivalent outcome of the static game of complete information can be obtained under incomplete information by a process in which bidders follow a simple strategy, i.e. raising their bids slightly until it is no longer profitable, following the argumentation of (Edelman, Ostrovsky, Schwarz [2007]). In this case, the equilibrium is regarded as a stable rest point of natural bidder adjustment dynamics. In this section, we finally allow for repeated play and investigate collusion in an infinitely repeated generalized second-price auction. Some preliminaries are necessary. We will follow a model of infinitely repeated games according to (Mas-Colell, Whinston, Green [1995]). Let us assume there is an infinite number of discrete stages, and at each stage, players interact strategically in a one-shot fashion. In our context, an infinitely repeated generalized second-price auction is a tuple $\mathbb{G}^{B,GSP,\infty} = \{\mathbb{E}_P, \mathbf{V}, \mathbf{O}, \mathbf{B}, \psi, \gamma^t, (\delta_i)_{i \in N}\}$, where $\psi(\mathbf{b}) = (s(\mathbf{b}), \mathbf{q}^{GSP}(\mathbf{b}))$. Let $\overline{\mathbf{b}} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_t, \ldots\}$, $\mathbf{b}_t \in \mathbf{B}$ be a sequence of bids of players. The total utility of each player is the discounted sum of stage utilities, i.e. $U_i(\overline{\mathbf{b}}) = \sum_{t=1}^{\infty} \delta_i^{t-1} u_i(\mathbf{b}_t)$. In the following, we will search for a collusive scheme that constitutes a subgame-perfect Nash equilibrium in $\mathbb{G}^{B,GSP,\infty}$. In an infinitely repeated game, the space of Nash equilibrium strategies expands considerably. In the following, we assume that players use the service of a mediator in order to establish a successful cartel. Under complete information, perfect collusion can be obtained simply by applying the collusive mediator as defined in Proposition 3 at every stage of the game. Since using the service of the mediator constitutes a Nash equilibrium at any stage t, it constitutes a subgame perfect Nash equilibrium in the infinitely repeated game $\mathbb{G}^{B,GSP,\infty}$. However, in an infinitely repeated generalized second-price auction, bidders are able to establish a cartel even without the use of a mediator by playing a grim-trigger strategy, though the cartel is not optimal in the sense of the cartel that employs the collusive mediator. In particular, bidders submit bids equal to the mediator as defined in Proposition 3 with a slight modification: They bid infinitesimally higher than the valuation of the first bidder that is not assigned to a real position anymore. The explicit formula is obtained by replacing r in Proposition 3 by $v_{(s(\mathbf{v},i_c)+1)}$. Whenever a bidder deviates, the bidders play the VCG equivalent Nash equilibrium strategies in all stages of the game that still follow. Let us call this strategy profile COLLUSION. The following result is due to (VOROBEYCHIK, REEVES [2007]), assuming every T_j to be singleton:

PROPOSITION 6 *(VOROBEYCHIK, REEVES [2007]) The COLLUSION strategy profile is a subgame perfect Nash equilibrium if, for all players i,*

$$\delta_i \geq \max_{j\in K, k\leq j} \frac{(\alpha_k - \alpha_j)(v_{(j)} - v_{(m+1)}) - (\alpha_k(m-k) - \alpha_j(m-j))\varepsilon}{\alpha_k(v_{(j)} - v_{(m+1)}) - \alpha_j v_{(j)} - \alpha_k(m-k)\varepsilon + V_{sum}}, \tag{5.4}$$

where $V_{sum} = \sum_{k=j+1}^{m} v_{(k)}(\alpha_{k-1} - \alpha_k) + v_{(m+1)}\alpha_m$.

We omit the proof and refer to the original paper. The authors observe that collusion seems likely if the number of slots is below about 8, but will not be feasible anymore with even a few more slots since the bound on the discount factor δ approaches 1. By playing a stronger punishment strategy, such as playing the worst equilibrium strategy profile of bids as given in 3.12, the collusive agreement appears to be stronger since the bounds on δ decrease.

However, is a collusive agreement that implements an efficient allocation still possible under incomplete information in an infinitely repeated generalized second-price auction?

The answer is yes, in contrast to the results obtained in the case of static models. Again, (Vorobeychik, Reeves [2007]) present an explicit result. Still, players are required to reveal their valuations honestly in order to implement an efficient allocation. This can be achieved by letting the bidders play the VCG equivalent symmetric Nash equilibrium for as many stages T as necessary such that it is not profitable anymore to misrepresent valuations, and play the collusive scheme as stated above afterwards. In the sense of the thesis at hand, a mediator is designed that acts in the sense of the VCG mediator from Proposition 2 for T stages, and in the sense of the collusive mediator from Proposition 3 afterwards. However, since there are T stages in which bidders are required to play a noncooperative equilibrium, the collusive scheme is far from optimal. Furthermore, the agreement is very limited since it only holds if player's discounted maximum payoff from the collusion game does not exceed the ratio of a player's click-through rate in his current position, say j, and the click-through rate of position $j + 1$. Hence we regard the collusive agreement to be not satisfactory at all.

More promising approaches are those of (Aoyagi [2003]) and (Feng, Zhang [2007]). The former shows that collusion is possible through inter temporal payoff transfers that serve as substitutes for monetary side payments. He presents a *bid rotation scheme* in a model of two bidders that engage for a single product in an infinitely repeated auction of the same format at any stage. It can be shown that by playing asymmetric equilibrium strategies, players can achieve even higher profits than in a bid rotation scheme that employs identical bids, which is shown to be optimal for a variety of parameters by (Athey, Bagwell, Sanchirico [2004]). In particular, (Aoyagi [2003]) develops a collusive scheme that makes use of a mediation device in terms of (Myerson [1986]) who implements the respective asymmetric equilibrium strategies. However, it can be easily thought of a mediator in terms of the thesis at hand that does the job: The mediator uses a random device in order to play on behalf of the players. In a symmetric phase S, the mediator implements the efficient allocation by bidding the reserve price on behalf of the player who reports a higher valuation $\hat{v}_i$ and zero for the other player.

More formally, we get for every player i and every stage $t \in S$,

$$\mathbf{m}^S(\mathbf{v}) = \begin{cases} r & \hat{v}_i \geq \max\{r, \hat{v}_j\}, \\ 0 & \text{otherwise.} \end{cases}$$

The ex-ante profit of the cartel associated with $\mathbf{m}^S$ is obviously optimal if the players report truthfully. However, this is not the case since every player has an incentive to overstate his valuation. Therefore, there is an incentive compatible second phase $A_i (i = 1, 2)$ to which the mediator switches with positive probability. It disfavors the bidder that reported a higher valuation by promising to him a lower continuation payoff. In particular, the mediator bids on behalf of player i (who reported a higher value in phase S) the reservation price if his valuation exceeds the reservation price and bidder j's valuation does not; and zero otherwise. It can be shown that phase A_i is incentive compatible. It will last for m stages before play returns to phase S. In fact, (Aoyagi [2003]) proves that the bid rotation scheme is incentive compatible and generates strictly higher profits for the bidders than any noncooperative equilibrium for sufficiently patient players. Note that the mediator overcomes the problem of parasite bidders, since a bidder who would overstate his valuation by pretending to have a valuation that is higher than the reserve price will earn negative profits at some stages of the game. Since there are stages in which another bidder than the one with the highest valuation obtains the product, the cartel cannot extract the full surplus anyway. A prevalent deficiency remains in the model of (Aoyagi [2003]). Such a more sophisticated mechanism requires a considerable amount of communication. However, the logic underlying the bid rotation scheme is striking and simple: Today's winners compensate today's losers by deferring to them in later stages.

The significant need of communication could be reduced by the use of a mediator in terms of the thesis at hand. One can think of a modified bid rotation scheme in the spirit of (Aoyagi [2003]) that exploits the generalized second-price auction and makes use of a mediator in addition. If the players are allowed to report to the mediator only once in advance of the legitimate auction, the game would effectively reduce to a static game. We leave the design of such schemes for future work. However, optimal collusion in which bidders extract the full surplus cannot be attained. This is in line with the findings of (Feng, Zhang [2007]). They find in a model of sponsored search auctions that advertisers will never form a perfect collusion since low-type bidders always have an incentive to overstate their valuations. They state that similar to the case in (Athey, Bagwell, Sanchirico [2004]), advertisers pay an "*information cost*" to reveal their true values associated with keywords through price wars. These results are finally in line with the findings of (Fudenberg, Levine, Maskin [1994]), who show that efficiency results are only available in independent private value models with *finite* signal. Obviously, players' types are not finite in our setting.

Let us conclude this chapter by shortly mention how sellers may react as a response to possible collusion. A well-studied option is to choose an appropriate reserve price. In repeated auctions, the corresponding treatment would be to choose the reservation price as a function of game history. Recent work of (Che, Kim [2009]) suggests that the auctioneer can retain any revenue that is feasible without collusion by prescribing a non-trivial probability of not selling the product to collusive bidders.

6 Variations

6.1 Asymmetric Equilibria

In our analysis of equilibrium behavior in Chapter 3 we focused on a particular subset of equilibria, i.e. symmetric Nash equilibria. These equilibria are symmetric in the sense that every player i expects to pay the same price for a given position j. But notice that we ignored the existence of a certain form of asymmetry in bidders' incentives to bid for a given position. Whenever a bidder wants to be assigned to a position $j > s(\mathbf{b}(\mathbf{v}), i)$, he just has to underbid the price he pays, i.e. the bid of the player assigned to position j and finally pays $b_{(j+1)}$, while he has to overbid the bid of a player assigned to position j for every $j < s(\mathbf{b}(\mathbf{v}), i)$ and hence pays $b_{(j)}$. Thus, we concentrated on a subset of equilibria in which bidders not even have an incentive to win a position $j < s(\mathbf{b}(\mathbf{v}), i)$ if they are only charged a price of $b_{(j+1)}$ rather than $b_{(j)}$. Hence, we provide a more general definition of equilibrium prices in the game $\mathbb{G}^{C,GSP}$.

DEFINITION 11 *A Nash equilibrium set of prices in the game* $\mathbb{G}^{C,GSP}$ *satisfies*

$$\alpha_j(v_{(j)} - p_j) \geq \alpha_k(v_{(j)} - p_k) \quad \forall\, k > j \tag{6.1}$$

$$\alpha_j(v_{(j)} - p_j) \geq \alpha_k(v_{(j)} - p_{k-1}) \;\forall\, k < j \tag{6.2}$$

where $p_j = b_{(j+1)}$.

(VARIAN [2007]) finds that the lower bound of Nash equilibrium prices is smaller or equal the lower bound of symmetric Nash equilibrium prices. The upper bounds are equivalent, which is not surprising since the conditions that restrict equilibrium bids on the upper bound are the same in both definitions. Hence the same holds for the revenue of the search engine. (Borgers, Cox, Pesendorfer, Petricek [2007]) show that there exist asymmetric equilibria that yield inefficient allocations. This is a contrary result to the findings about properties of symmetric Nash equilibria. They provide a modified vector of symmetric Nash equilibrium bids that constitutes an asymmetric equilibrium such that the positions of the players assigned to the two highest positions are exchanged. This is reached in particular by making the bidder in the former position two of the symmetric Nash equilibrium submit a very large bid, and the bidder in former position one to bid infinitesimally higher than the bidder in position three. Moreover, they show that for the case of $N = K = 3$, there exist asymmetric equilibria in which any player i can win any position $j \in K$ if certain conditions on marginal values of bidders hold. The authors judge these conditions to be "*very weak*". Hence they question the focus on symmetric Nash equilibria. In particular, they show that the asymmetric Nash equilibria that they present cannot be ruled out by means of neglecting Nash equilibria in weakly dominated strategies. They provide explicit (and relatively wide) bounds on bids that are not weakly dominated. However, they restrict attention only to the case were the dominating strategy is a pure strategy.

A quite striking, practical argument for the selection of symmetric Nash equilibria remains. The *generalized English auction* that (Edelman, Ostrovsky, Schwarz [2007]) introduced in order to illustrate the process by which bidders may coordinate to the VCG equivalent outcome, i.e. by raising their bids slightly until this is no longer profitable.

6.2 Mediators in General Position Auctions

In (Ashlagi et al. [2008]), the authors present a characterization of a group of position auctions for which a mediator exists that is able to implement φ^{VCG}. Additionally, they show an impossibility result for self-price position auctions, i.e. position auctions in which each bidder has to pay his bid. We will shortly sketch their very general findings and present a mediator for the *generalized third-price position auction (GTP)* that we construct due to the authors conventions. We omit all proofs and refer to the original paper. Observe the following preliminaries by (Ashlagi et al. [2008]) before we finally present a mediator for the GTP position auction:

Definition 12 *(GLP Position Auctions) A position auction $\mathbb{G}$ is a generalized lower price auction, if the payment of each player who is assigned to a position in K is a function of the bids of players assigned to "lower" positions (i.e. positions with higher indices). More specifically, for every $j \in K$ and for every two bid profiles $\mathbf{b}^1, \mathbf{b}^2 \in \mathbf{B}$ such that $b^1_{(l)} = b^2_{(l)}$ for every $l > j$, $p_j(\mathbf{b}^1) = p_j(\mathbf{b}^2)$.*

Definition 13 *(VCG Cover) A position auction $\mathbb{G}$ is a VCG cover if for every $\mathbf{v} \in \mathbf{V}$ a vector of bids $\mathbf{b} \in \mathbf{B}$ exists such that $\psi^G(\mathbf{b}) = \varphi^{VCG}(\mathbf{v})$, where $\psi^G(\mathbf{b}) = (s(\mathbf{b}), q(\mathbf{b}))$.*

Definition 14 *(Monotonicity) A position auction $\mathbb{G}$ is monotone if $p_j(\mathbf{b}) \geq p_j(\mathbf{b}')$ for every $j \in K$ and for every $\mathbf{b} \geq \mathbf{b}'$, where $\mathbf{b} \geq \mathbf{b}'$ if and only if $b_i \geq b'_i$ for every $i \in N$.*

Proposition 7 *(Ashlagi et al. [2008]) In a position auction $\mathbb{G}^B$, an individually rational mediator exists that implements φ^{VCG} in $\mathbb{G}^B$ if the following three conditions hold:*

(a) 1. *$\mathbb{G}^B$ is a GLP position auction.*

   2. *$\mathbb{G}^B$ is a VCG Cover.*

   3. *$\mathbb{G}^B$ is monotone.*

(b) The set of conditions 1-3 is minimal. If any of the conditions 1-3 is dropped, one can construct a position auction which satisfies the two other conditions, but φ^{VCG} cannot be implemented by an individually rational mediator.

While it is comparatively easy to verify if a position auction is monotone and if it holds that it is a GLP position auction, it is sometimes more involved to verify that an arbitrary position auction possesses the VCG cover property. Therefore, the authors provide some characteristics that are necessary for a position auction to be a VCG cover. These imply the payment function $p_j(\cdot)$ to be continuous as well as some monotonicity requirements of $p_j(\cdot)$. We will not present these characteristics in detail and refer to the original paper since it is not essential to capture the idea of the following findings. In order to proof that the set of conditions is minimal, the authors show that whenever a position auction is a VCG cover and either monotonicity or GLP property but not both conditions hold, assuming that a mediator exists that implements VCG by truthful mediation yields a contradiction. If a position auction is not a VCG cover, no mediator can implement the VCG outcome by truthful mediation anyway. Let us now exemplarily present a mediator for the *generalized third-price position auction* that implements the VCG equivalent outcome, i.e. $\varphi^{\mathbf{m}} = \psi^{VCG}(\mathbf{v})$:

PROPOSITION 8 *(VCG Mediator for GTP) Let $n \geq m+2$. The following mediator implements φ^{VCG} in a third price position auction:*

(a) For every $\mathbf{v} \in \mathbf{V}$ let $\mathbf{m}_N(\mathbf{v}) = \mathbf{b}(\mathbf{v})$, where $\mathbf{b}(\mathbf{v})$ is defined as follows:

- $\mathbf{b}_i(\mathbf{v}) = p^{VCG}_{s(\mathbf{v},i)-2}(\mathbf{v})$ *for every player i such that $3 \leq s(\mathbf{v}, i) \leq m$.*
- $\mathbf{b}_i(\mathbf{v}) \in \left[p^{VCG}_m(\mathbf{v}), p^{VCG}_{m-2}(\mathbf{v})\right)$ *for the player assigned to position $m+1$, i.e. $s(\mathbf{v}, i) = m+1$.*
- $\mathbf{b}_i(\mathbf{v}) = p^{VCG}_m(\mathbf{v})$ *for the player assigned to position $m+2$, i.e. $s(\mathbf{v}, i) = m+2$.*
- $\mathbf{b}_i(\mathbf{v}) = \frac{p^{VCG}_m}{1+\rho}$*, for every player i such that $s(\mathbf{v}, i) > m+2$, and for some arbitrary but fixed $\rho > 0$.*
- $\mathbf{b}_i(\mathbf{v}) = \epsilon + \eta + p^{VCG}_1(\mathbf{v})$ *for the player assigned to position 1, i.e. $s(\mathbf{v}, i) = 1$, and for some arbitrary but fixed $\epsilon > 0, \eta > 0$.*
- $\mathbf{b}_i(\mathbf{v}) = \epsilon + p^{VCG}_1(\mathbf{v})$ *for the player assigned to position 2, i.e. $s(\mathbf{v}, i) = 2$ for some arbitrary but fixed $\epsilon > 0$.*

(b) Let $\epsilon > 0$ be fixed. For every i and for every $\mathbf{v}_{-i} \in \mathbf{V}_{-i}$, let $\mathbf{v}^i = (\mathbf{v}_{-i}, M(\mathbf{v}_{-i}))$, where $M(\mathbf{v}_{-i}) = \epsilon + max_{j \neq i} v_j$. $\mathbf{m}_{N \setminus \{i\}}(\mathbf{v}_{-i}) = \mathbf{b}_{-i}(\mathbf{v}^i)$ for every $i \in N$ and every $\mathbf{v}_{-i} \in \mathbf{V}_{-i}$.

(c) For every $S \subset N$ such that $1 \leq |S| \leq n-2$, $\mathbf{m}_S(\mathbf{v}_S) = \mathbf{v}_S$ for every $\mathbf{v}_S \in \mathbf{V}_S$.

Proof. Analogous to the proof in Ashlagi et al. [2008] it can be shown that $\varphi^m = \psi^{VCG} = (s(\mathbf{v}), \mathbf{q}(\mathbf{v}))$. Furthermore, the mediator described above is constructed according to the restrictions of *Mediator 2* in Ashlagi et al. [2008], where it is proven that such a mediator implements φ^{VCG} by truthful mediation under certain conditions, which emerge to hold for the given setting. □

Let us conclude by shortly illustrating the idea behind the mediator above. Obviously, given all other players report truthfully to the mediator, reporting a wrong value to the mediator cannot be profitable for any player $i \in N$ since VCG is truthful. If all players but i use the service of the mediator, the mediator pretends that the player who deviates has the highest among all values and submits a vector of bids $\mathbf{b}_{-i}(\mathbf{v}^i)$ accordingly. There exist some value $\tilde{v}_i \neq v_i$ that yields the same profit for player i when reported to the mediator than participating directly and bidding b_i. But since VCG is truthful, this cannot be profitable. Finally, it can be shown that the VCG mediator for the generalized third-price position auction is individually rational since by submitting a vector of bids $\mathbf{b}_{-i}(\mathbf{v}^i)$ and by the first part of Lemma 2, he can still guarantee each bidder who uses his service a profit greater than or equal zero.

6.3 Position Auctions with Quality Factors

In practice, companies like Google apply a slight variation of position auctions as modeled in this thesis in order to sell ads. They introduce quality factors, i.e. preferences over players. We already introduced the idea in Section 2.1 under the term of *relevance*. Such preferences express the probability that an ad of a certain advertiser is clicked if it is shown in a certain position. This makes perfect sense from the point of view of the search engine, since the search engines goal is to maximize revenue, i.e. the number of paid clicks on ads. Formally, we omit the assumption that the click-through rate for any position j does not depend on the identity of a player. Therefore, let us denote by $\beta_i > 0$ a fixed quality factor for every player $i \in N$. The click-through rate for any bidder i assigned to position j therefore becomes $\varsigma_j^i = \beta_i \alpha_j$. Let $\beta = (\beta_1, \ldots, \beta_n)$ be the vector of quality factors. Let $\mathbb{G}(\alpha, \mathbf{p})$ denote an arbitrary position auction. Now if quality factors are introduced for each player, we define the new auction with quality factors by $\mathbb{G}^Q(\beta, \alpha, \mathbf{p})$.

Note that, if we assume that the click-through rate of any position j depends not only on the position but also on the identity of the player, the former efficient position auction that ranks players by their bids is not efficient anymore. Let us demonstrate this finding by a simple example. Therefore, assume there are two players i, j who are assigned to a position in K respectively. Let the valuation of player i be $v_i = 8$ and $\beta_i = 1$. For player j let $v_j = 14$ and $\beta = 1/2$. Now since the allocation rule s of a position auction without quality factors is consistent with the vector of valuations, player j is assigned to a position $s(\mathbf{b}, j) < s(\mathbf{b}, i)$. However, the revenue of player i per click equals $v_i \cdot \beta_i = 8$, while the revenue of player j is only $v_j \cdot \beta_j = 7$. But efficiency requires player i to be assigned to a better position than player j. In $\mathbb{G}^Q$, players are ranked according to the vector of adjusted bids $\beta\mathbf{b} = (\beta_1 b_1, \beta_2 b_2, \ldots, \beta_n b_n)$. Let us denote the resulting allocation by $s_\beta(\mathbf{b}) = s(\beta\mathbf{b})$. A position auction with quality factors ranks players efficiently. We will omit the proof since it is very similar to the proof of Observation 1.

Furthermore, let $i(\beta\mathbf{b}, j)$ be the player that is assigned to position j, i.e. $s_{i(\beta\mathbf{b}, j)} = j$. Let $p_j^i(\beta\mathbf{b}) = b^{i(\beta\mathbf{b}, j+1)} \beta^{i(\beta\mathbf{b}, j+1)}$. A player i assigned to position j, i.e. $s(\beta\mathbf{b}, i) = j$ has to pay $\frac{1}{\beta_i} p_j(\beta\mathbf{b})$ since by construction and in order to receive position j it must hold that $b_i \beta_i \geq b^{i(\beta\mathbf{b}, j+1)} \beta^{i(\beta\mathbf{b}, j+1)}$. Now note that we can express every position auction with quality factor as a position auction without quality factors if we redefine the valuation of every player to be the product of his quality factor and his valuation, i.e. $b_i \beta_i$. To see that this is true, observe that we can write the symmetric Nash equilibrium requirement for every player i as stated in Definition 3.1 for a position auction with quality factors as

$$(v_{(j)} - \frac{1}{\beta_i} p_j^i(\beta\mathbf{b}))\beta_i \alpha_j \geq (v_{(j)} - \frac{1}{\beta_i} p_k^i(\beta\mathbf{b}))\beta_i \alpha_k \quad \forall\, k \in K.$$

But this can be simply written as

$$(\beta_i v_{(j)} - p_j^i(\beta\mathbf{b}))\alpha_j \geq (\beta_i v_{(j)} - p_k^i(\beta\mathbf{b}))\alpha_k \quad \forall\, k \in K.$$

Now the same logic as in Chapter 3 can be applied. Let us call every position auction with quality factors a $\beta-$position auction. Let the $\beta-$VCG position auction be denoted by $\mathbb{G}^{Q,VCG}(\beta, \alpha, \mathbf{p}^{VCG})$. By using a result of (ROBERTS [1979]), (ASHLAGI ET AL. [2008]) state that "*the $\beta-$ VCG position auction chooses the allocation of a weighted VCG mechanism with a vector of weights β, and the payment of a player equals the standard weighted VCG payment.*" But since a position auction with quality factors is not different from a position auction without quality factors with an adjusted vector of valuations $\beta\mathbf{v}$, and since we proved that for every vector of valuations $\mathbf{v} \in \mathbf{V}$, an individually rational mediator exists that implements the VCG outcome function in $\mathbb{G}(\alpha, \mathbf{p})$, there must exist an individually rational mediator in $\mathbb{G}^{Q}(\beta, \alpha, \mathbf{p})$ that implements the $\beta-$VCG outcome function. In particular, the $\beta-$version of the VCG-mediator does the job. All results on collusive mediation devices still hold, as the case may be in adjusted versions.

6.4 Utility Symmetry

We define another type of anonymity and show that the results obtained in Section 5.2 hold for this notion as well. We say that an allocation rule is utility symmetric, if for any two players who report their types truthfully and exchange bids and valuations, it holds that they will end up with the utility of the other player respectively. The formal definition is taken from (ASHLAGI [2008]):

DEFINITION 15 *A position auction is called utility symmetric if for every two distinct players i, j, for every vector of bids of the other players $\mathbf{b}_{-(i,j)}$ and for every v_i, v_j it holds that*

$$u_j(v_i, \mathbf{b}^{ij}) = u_i(v_i, \mathbf{b}),$$

where $\mathbf{b} = (v_i, v_j, \mathbf{b}_{-(i,j)})$.

Utility symmetry is not equivalent to the notion of an anonymous allocation rule. One can construct position auctions that have an anonymous allocation rule, but are not utility symmetric and vice versa. (ASHLAGI [2008]) shows that "*every truth-revealing* *utility symmetric position auction is a VCG position auction.*"

In the proof he makes use of the following lemma that goes back to (Myerson [1981]) for the case of a position auction with a single position, i.e. $m = 1$, which was extended to general position auctions by (Archer [2004]):

Lemma 8 *(*Myerson *[1981]) If a position auction* $(s, \mathbf{p})$ *is truth-revealing then for every* $v_i \in V_i$ *and every* $\mathbf{b}_{-i} \in \mathbf{V}_{-i}$*, it holds that*

$$u_i(v_i, (v_i, \mathbf{b}_{-i})) = u_i(0, (0, \mathbf{b}_{-i})) + \int_0^{v_i} \alpha_{s_i(x, \mathbf{b}_{-i})} dx. \tag{6.3}$$

It states that the expected payoff of a bidder in a truth-revealing direct position auction depends only on the allocation rule s, and that for any two truth-revealing position auctions with the same allocation rule s, the payoff function for each bidder differs by at most an additive constant, which is equal to the utility of the worst possible type.

Proof. Lemma 8 states a by now well-known property of truth-revealing mechanisms. A short but well-arranged review is given in the chapter on mechanism design in (Krishna [2002]). For more details, we refer to the original paper respectively. □

(Ashlagi [2008]) shows in a similar manner as in the proof for an anonymous allocation rule and by use of Lemma 8 above, that a bid profile for which a non-highest bidder would receive a higher position than a highest bidder contradicts utility symmetry. Therefore, the allocation rule s of a truth-revealing utility symmetric position auction $(s, \mathbf{p})$ must be a welfare maximizer and hence by Holmstrom's Lemma it must be a VCG position auction. Thus, the results from Section 5.2 extend to the case of utility symmetric position auctions.

7 Conclusion

In the thesis at hand, a mechanism is presented that is the most frequently used mechanism in the Internet advertising market today and generates billions of dollars of annual revenues. In its current version, it was introduced not earlier than in February 2002. The so-called *generalized second-price auction* places links of advertisers to positions on a screen is descending order of their bids, and every advertiser is charged the bid of the advertiser who is ranked only one position below. We find that, although the mechanism resembles the VCG mechanism, it is not strategically equivalent. In particular, it lacks too many of the desirable properties of a VCG auction, such as, truth-telling to be a weakly dominant strategy or an equilibrium in dominant strategies. Nevertheless, we show in a static model of complete information that among the set of symmetric Nash equilibria, the most profitable equilibrium outcome for the bidders and hence the worst equilibrium outcome for the search engine is equivalent to the equilibrium outcome of a corresponding position auction that is designed according to the rules of VCG. Though the complete information framework is far from perfect, the results obtained serve as an important estimate of equilibrium behavior under incomplete information. It is assumed that bidders experiment with their bids over time and thereby gather enough information to conduct other bidders' valuations. Furthermore, due to the highly dynamic structure of the auction—bidders can change their bids whenever they want—bids in any stable equilibrium must be a static best response to each other. In fact, empirical results by (VARIAN [2007]) suggest that the actual behavior of bidders can be explained approximately well by the results obtained from a static model of complete information. Moreover, we find that bidders can coordinate to the bidder-optimal outcome by the use of a mediation device, even if the vector of valuations is not common knowledge. We present a formal definition of mediators and apply it to the context of position auctions. A mediator in terms of this thesis is a reliable entity that can bid on behalf of the advertisers in the auction, but bidders are free to choose whether to use the service of the mediator or participate in the auction directly. We design a mediator that coordinates bidders to a collusive outcome in a static model of complete information in which bidders only pay the reserve price plus some minimum increment. In equilibrium, all bidders profit from using the service of the mediator.

We show an impossibility result under incomplete information. Efficiency implies non-cooperative profit levels—even by the use of a mediator—in any position auction with incomplete information. We find that even if we equip the mediator with the ability to facilitate transfer payments among bidders, no efficient collusive protocol exists. This is due to the fact that profitable collusion attracts parasite bidders who only participate in the auction to get a share of the cartel's profits and hence, transfer payments are no longer an option. Finally, we extend our model and allow for repeated play. We present a collusive scheme that constitutes a subgame-perfect Nash equilibrium in an infinitely repeated generalized second-price auction. Nevertheless, firms cannot form a perfect cartel. Through price wars, they pay a significant *information cost* in order to learn about each other's types before they can successfully establish an efficient cartel. The relative robustness to collusion that we observe is a remarkable feature of a mechanism that developed as a result of the evolution from inefficient mechanisms that were gradually replaced by superior designs. Despite its simplicity and the existence of stable equilibria, this seems to us a decisive factor for its immense commercial success.

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Erklärung zur Urheberschaft

Hiermit erkläre ich, dass ich die vorliegende Arbeit allein und nur unter Verwendung der aufgeführten Quellen und Hilfsmittel angefertigt habe.

Kai Becker

Berlin, June 1, 2009